走向代数表示论

刘绍学文集

LIU SHAOXUE WENJI

北京师范大学出版社

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▶ 1956年6月8日，在莫斯科大学刚答辩完副博士论文后摄于校园内的一个花坛边



▼ 1991年刘绍学和他的学生们首次秋游，〔左起：王志玺、章璞、刘绍学、肖杰、林亚南、彭联刚〕



▼ 1991年10月，在参加广西师范大学举行的第一次中日环论国际会议时摄于桂林漓江的游船上。(第一排左起：林亚南、张英伯、刘绍学、C. M.Ringel、章璞、惠昌常、彭联刚；第二排：郭晋云)



▶1994年8月，在南开大学举行中国代数年时摄于南开大学数学研究所楼前。〔第一排左起：C.M.Ringel、惠昌常、邓邦明、张顺华；第二排左起：刘绍学、郭晋云、M. Auslander、I.Reiten、张英伯、张跃辉；第三排左起：李思泽、张卫、吴求先、姚海楼、杜先能〕





◀ 1993年学生们祝贺刘绍学64岁生日，〔左起：朱彬、杜先能、刘绍学、李思泽、张顺华、黄兆泳、姚海楼、张跃辉〕

▼1999年，学生们为刘绍学先生过70岁生日摄于北京师范大学。〔第一排左起：肖杰、张英伯、刘绍学、肖元正、曹文龄、罗运纶、郭晋云；第二排左起：彭联刚、李思泽、林亚南、惠昌常、朱彬、章璞、王志玺、周梦、刘海霞〕

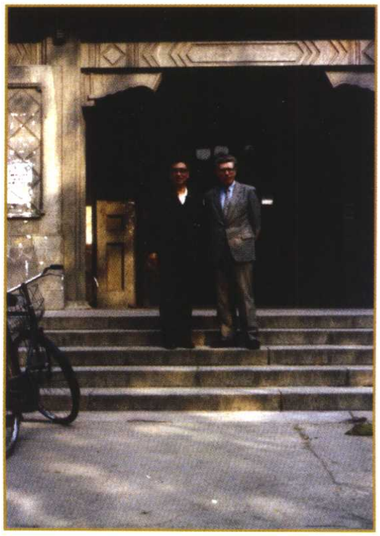






▲ 1999年10月，在北京师范大学举行全国代数会，于天伦王朝饭店的晚宴上还为曹锡华、周伯壎、刘绍学三位先生祝寿。〔图为刘绍学在晚宴上助兴〕

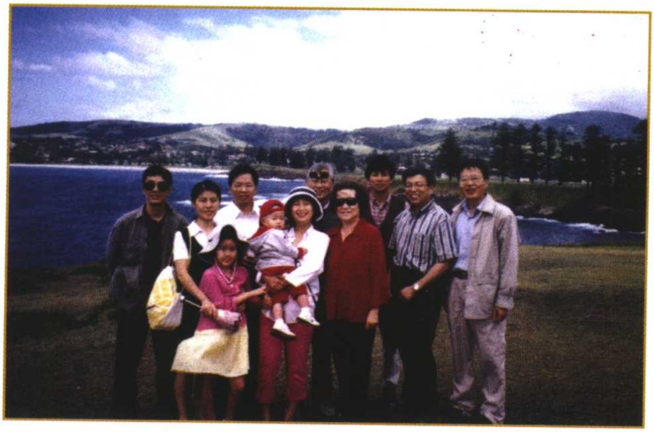
▲ 1999年5月，北京师范大学数学系《五教授执教五十周年庆祝会》上向刘绍学等献花〔右起:刘绍学、王梓坤〕



▶ 1999年10月，俄罗斯科学院院士、莫斯科大学代数教研室主任A.I.Kostrinkin应刘绍学邀请来北京师范大学讲学期间，摄于北京师范大学数学楼前。



◀ 2000年6月访问比利时期间，刘绍学一家人在Antwerp大学教授F.Van. Oystaeyen家做客。曹文龄摄〔自右逆时针转：刘绍学、Fred、其夫人、其女儿、其儿子、刘绍学之大女儿刘艺冰〕



◀ 2001年10月，刘绍学夫妇去澳大利亚探亲，摄于卧龙岗附近海边。〔第一排左起：徐文、刘路、其女儿徐堪、华九召夫人及其子、曹文龄、杜杰、彭联刚；第二排左起：华九召、刘绍学、邓邦明〕

▼ 2004年10月6日，在扬州大学举行的《全国代数表示论高级计论班》的晚宴上祝贺刘绍学75岁诞辰〔站立讲话者是郭晋云，其前面左起是曹文龄，刘绍学，王宏玉（扬州大学数学科学学院院长）〕



序　言
Preface

On the occasion of the 75th birthday of Professor Shao-xue Liu, Beijing Normal University is publishing a volume containing a collection of his papers. In this preface, I want to focus the attention to his initiative to study problems in representation theory. I will outline the development of a very fruitful cooperation between China and the West concerning the representation theory of finitedimensional algebras, along the line of our mutual contacts. During the years, we became true friends, and I will add some personal comments about my introduction to China and the Chinese culture. Professor Shao-xue Liu has to be considered as the Nestor of algebra in China and we are very happy that he shifted the interest of many of his students and collaborators to algebras and their representations. Indeed, he is the founder of a large and very successful Chinese school in representation theory of finitedimensional algebras which covers a broad range of topics and deals with many different applications.

This school has its origin at BNU and now has strong outlets at many other scientific institutions not only in Beijing (one has to mention at least Tsinghua University and the Academy of Science), but also Shanghai, Chengdu, Hefei, Xiamen, Changsha, and there are many more places.

It does not seem to be necessary to comment on the individual papers in this collection, but one should look also at Liu's book on Rings and Algebras which had and still has a tremendous influence. As the title indicates, it is the structure theory for rings and algebras which is the central theme, questions concerning representations are treated only marginal (still, it deals with semisimple modules and the corresponding double centralizer property). In particular, the Wedderburn-Malcev theory, as developed in the book, has to be considered also as starting point for dealing with representations of algebras. The book has served as a basic reference not only for his own students, but seems to have been used throughout China. I tried to persuade Springer to publish a translation into English because it could have been an important and very useful addition to the Western market, but the publishing company did not want to get competition to its other books addressed to the same audience.

1985

In 1985, Professor Liu visited several European universities, including Bielefeld—this was our first contact. The aim of his travel was to get in touch with some of the active research centers in the West dealing with rings and modules. He had carefully analyzed the present state of research in algebra. Apparently, he was impressed by the stormy development of the representation theory of finitedimensional algebras and he had decided that China should get involved in such investigations. He wanted to start to cooperate with all the relevant schools, hoping that after some time the Chinese algebraists would be able to participate in the worldwide research competition. It was obvious that he had a clear vision of the role China should play in the future: to become a leading contributor to the research.

It was August, the typical holiday time in Germany, so not many

mathematicians were around, and usually no lectures are scheduled at this time. Still I felt that I should ask Professor Liu whether he would like to present some results, but with the warning that the audience should be rather small. Yes, he said, he would be happy to give a lecture, or better: to give four short lectures, on four different topics, each one of 15 minutes. I was uncertain that this could work, but what could I do? The lecture given by him was a real surprise and it was very well arranged: It was the precision of his thinking, the perfect presentation and the well-chosen motivation which impressed us very much. In any one of the four parts, he first outlined in detail the context, with all the necessary definitions, specified a problem and finally showed the solution (or a partial solution) which had been obtained by him and his students during the last years. In this way we obtained an impressive insight on the scientific interests and the research power of his working group. The broad range of topics was remarkable: T-ideals, nilpotency of radicals, algebras of finite representation type, and non-associative rings (Jordan algebras). At that time, Skowronski (from Torun, Poland) was also in Bielefeld, and the part which was of most interest to Skowronski and me was the third part, dealing with algebras of finite representation type (Xiao had shown that the module category of such an algebra is always perfect). I still remember that after the lecture there was a long (and fruitful) discussion on possible generalizations and on related questions such as the pure-semisimplicity conjecture. In this way, Liu's lecture has to be considered as the starting point for a long cooperation.

This was his plan: he wanted to send students to all the relevant universities in the world in order to obtain there a Ph.D., and conversely to invite some of the experts for a stay at BNU in order to give series of lectures. As first measure, Deng was sent to Zurich where Gabriel was working, Luo to Carleton University (Dlab), Shiping Liu to Liverpool (Brenner and Butler), and Chang-chang Xi to Bielefeld. At that time, M. Auslander who was also approached saw problems to accept a student, but later Luo moved to Brandeis University. After my visit to China in 1987, Ying-bo Zhang

came as second Ph.D. student to Bielefeld and then later Ya-nan Lin as a third one. Furthermore, Guo obtained a Humboldt fellowship for working with me. Later several other Humboldt fellowships were awarded to members of the Chinese representation theory group (to Pu Zhang, Lian-gang Peng and to Bang-ming Deng). This, of course, increased the mutual contacts between Germany and China considerably.

Most of the students mentioned had been for some time at BNU, but actually they were selected from all over the country. In fact, it seemed to me that Professor Liu and Professor Cao (from ECNU, the East China Normal University) felt their responsibility to care for all the young Chinese algebraists, at least those studying at "Normal Universities" (those devoted to the training of future teachers). The best of these students were invited to come to BNU or to ECNU, to get there their education, so that later they could go back to their home university with a proper training. And some of these were selected to obtain a Ph.D. from abroad, or they were asked to apply for other support such as a Humboldt fellowship.

Shao-xue Liu had brought several photos with him in order to present a more vivid picture of BNU, and it was clear that he was proud of his university. Most of the pictures were very convincing, so for example I learned that Xun Lu had been a member of BNU (at that time, a nice volume with some of the most famous stories and texts of him had appeared as a volume of Die Andere Bibliothek). But there was one picture which irritated us. It showed the gate of BNU: it neither looked old (that would be a convincing argument in the West for presenting it), nor very fancy. Why does someone care about a gate? Are there actually still walls around a university or other institutions (indeed, these are!)? And could it be that the doors are closed in the night, as it was customary in the middle age in European towns? It took me quite a while (and several trips to China) to get some understanding of the importance of the walls around a danwei. A university in China is more than a place to study and to do research. Professors and students, as well as all the subsidiary staff do not only work together but really

live together, in a well-defined area, surrounded by a wall. In some sense, such a university also resembles a monastery, well-shielded from the outside, with nice gardens and not too many distractions. These units (which may be a university, or a factory, or some administration…) have their own schools, their hospitals, their shops and restaurants, they are nearly self-sustaining, something like small towns in themselves. A city such as Beijing breaks up into thousands of independent units—and the gates symbolize the importance of the units.

During Liu's stay at Bielefeld, we discussed various topics, ranging from mathematics to culture and politics. It was of interest to see his rating of different mathematical subjects, and his clear assessment of the relevance of notions and results. For example, I remember that he raised the question why I follow the tradition to speak of "torsion theories"? Is this a theory? Or is it not just a typical mathematical object, namely a pair of subcategories, and thus should be named "torsion pair"? And he was right. So, in later publications I changed the terminology.

1987

My first visit to China was in March and April 1987, being invited by BNU and ECNU. The German research council sponsored the flight and I remember that my application was commented by one of the officials by saying that recently there were many similar applications: Apparently, it is easier to invite lecturers than to buy books.

My lectures were scheduled in the morning, usually for three or four hours. Three days lecture were followed by a free day. The graduate students who attended my lectures were Guo, Xiao, Ying-bo Zhang and Tang, thus two male and two female students, and there were around ten undergraduate students, also here the proportion of female students was high. This was quite strange for me, since in Germany mathematics is considered in public as a "male" discipline, and unfortunately female students are often scared away. In later years, it seemed to me that a similar tendency was observed in China — a very unfortunate development. All the students were

very active, some came in the afternoon or even late in the evening to ask questions and to discuss with me partial solutions of their home work. And they were very well prepared for my lecture, even showing me copies of original papers related to the content of my teaching. They had seen already quite a lot of the relevant definitions and had read theorems and proofs, however they had not worked through any example at all, so they had severe difficulties to see the relevance of the results.

My lectures followed notes which I had made for a similar series of lectures at Antwerp, however the reaction of the audience was quite different: the atmosphere at BNU was very enthusiastic, but I had to spend much more time on some of the details when dealing with specific applications. So in later lectures, I changed the presentation and shifted all the attention to a better understanding of relevant examples. It was clear that it was important for Chinese students to get personal instructions, not just to obtain books to be read.

One has to recall the hardship at that time: it was very cold, the students were wrapped in heavy coats in order to survive. When giving lectures, I was happy of being able to move around, and I would have been afraid to sit there for three or four hours (also, as lecturer, I had the privilege to obtain all the time hot tea, a very kind gesture). It was end of March, and the official winter period was over (the weather did not care about that), so there was no heating anywhere. An exception was made in the building for foreign experts (where I stayed), and for Professor Tuan whom we visited at Beijing University. He was a student of R. Brauer and it was of interest to listen to his recollection.

To work with Chinese students was usually a pleasure. But sometimes there were surprises. I remember that Chang-chang Xi, being new at Bielefeld, once contacted me claiming that he did not understand some proof of mine (in the Springer Lecture Notes Volume 1099). Without looking at the corresponding pages, I tried to explain him the principles of the methods used; still he insisted that he had problems with the proof. So finally we

went through the text, line by line. When I mentioned in by passing that apparently there was a misprint, a t was typed instead of an n, he immediately responded: yes, if one uses there n instead of t, then the proof works well. Obviously, he never would have dared to believe that a written text could contain misprints!

And most of the Chinese students had a forceful desire to solve problems, if possible to solve a new problem every month. Such an attitude will not allow any polishing of proofs, no reflection on the best possible way for writting up the results. And I guess that Professor Liu himself, whose writting always was very refined, could not be pleased in this way.

During the first days of my stay, I had a cook for my own. He prepared every day a kind of Western dish just for me, but I wanted to have Chinese food. Thus, after some days, I took all my courage and copied some Chinese characters from my travel guide, telling him that I would not come back the next day. After that, I learned about the refectory for foreign students, where I enjoyed various kinds of Chinese food. But I was astonished to see the strict separation between Chinese and non-Chinese students (whether from Europe, America or say Vietnam) at that time. The food was fine, but of course it couldn't be compared to the jao-zi which I tasted when visiting the home of Professor Liu: they were prepared by his wife and were really delicious. Once, also Ying-bo Zhang invited me and cooked herself: I was allowed to watch her and to follow all the steps. Clearly, Chinese cooking is an art in itself. And there are many specialities like sea cucumbers, rice field snakes, or also dogs ("The yellow dogs taste best" —a sentence which brings every German table conversation to a sudden end), and astonishing variations of classical European ingrediants (for example potatoes cut into tiny sticks).

Shao-xue Liu also introduced me to Beijing operas. At that time, I did not even appreciate the European operas (in contrast to all kinds of other theater plays): They seemed to me too artificial, too far away from any reality (just imagine someone standing there and singing "I run away, I run

away?" Why doesn't he do what he asserts?), and the Chinese operas obviously looked even worse. The first one I saw was the White Snake, and this was a great shock for me. It took me some while to realize the essence of such operas: the perfectness of the presentation and the virtuosity, but also the symbolic reference, the refinement of style, the precise interplay between music and movement. Beijing operas were however popular only for elder people. Indeed, when I mentioned my interest to some of the students, they were surprised and told me: You are young, why do you care about such things? There was already concern about the survival of traditional operas, Guo accompanied me to a corresponding meeting. Coming back to Germany, I tried to learn more about operas, both the European and the Chinese ones. Yes, I have to admit that my understanding of the European operas relies on seeing Beijing operas.

After my visit to Beijing, I took a train in order to stay at ECNU for another week. On the way I wanted to stop over in order to climb the mount Tai Shan. Professor Liu accompanied me to Qufu (the home town of Kong Fu-Tse, and near the Tai Shan). However, it was not possible to preorder tickets, thus to continue for Shanghai, I had to stay at Qufu for three days. During these days, I gave some lectures at Qufu Normal University, apparently as one of the first visitors after the cultural revolution.

1991 and 1994

A further visit of me to China was planned for 1989. I had managed to make a booking for the Transsiberian Railway from Novosibirsk to Beijing, which was not easy at all-usually you had to use the Transsib all the way, but there was the Malcev conference at Novosibirsk which I wanted to use as a start. Quite a while after June, Ying-bo Zhang asked me whether I would cancel the trip to China. I replied: No, I will go. She said: You don't hesitate to go to China this year? I told her: No. She told me: You will not go. But I wanted. However, some weeks later, I obtained a letter from Professor Liu, telling me that he had heard that I had hesitations and that in this case I should not feel obliged to come even having made the

promise. I interpreted this as the message to cancel my plan——which I did with quite regret. At Novosibirsk, I met Professor Liu and found out that we both were unhappy about the cancelation of my visit, but is was too late to renew the arrangements.

I was excited to hear that there was the plan to hold a Chinese-Japanese ring theory conference in 1991. I liked the idea very much, because both countries have a long tradition in ring theory and I felt that despite of possible political concerns scientists should always cooperate. By that time I had strong contacts both to Japan and China, and so I was very pleased to be invited (as one of a small number of Westerners) to this conference, which was held at Guilin. The Chinese contributions showed in which way algebra had survived (and even flourished) during the cultural revolution, in complete isolation, without proper access to books or journals. Without much prerequisites, general properties of rings and modules had been studied. Most of the talks focused the attention to properties which hold for the vast majority say of rings, without any discussion of examples. Special properties of those rings which arise in nature, those which are of interest in other parts of mathematics, were not considered as being of greater interest. As a counter balance, I decided to speak on finitedimensional hereditary algebras, their properties and the relationship of this class of algebras to other parts of mathematics. My aim was to stress the importance of detailed studies of very concrete mathematical objects.

It is customary that during a mathematical conference one half day or even one day is reserved for some excursion. The organizers of the Guilin conference had planned even two such occasions: the visit of a marvellous stalactite cave and a boat trip on the river Li. The landscape around the river is very special, with mountains which look like sugar loafs. Some of these mountains are plain rocks, and looking at the stone one may envision hidden pictures. Liu challenged the participants to use their imagination and to explain what they see. For example, one of these peeks is famous for its "twelve horses". It was interesting to observe the different reactions to this

game. One of the mathematicians from Japan proposed instead to evaluate the phallic shape of the peaks, but Liu condemned this as too obvious and blatant.

Europeans are less enthusiastic about stones and the miracles they hide. Of course, the so-called precious stones are admired and are used for jewellery, and one finds stones like marble in entrance halls, as floor covering or table tops, but never you will take an individual stone as a decoration like a painting or a sculpture. I learned from Liu to follow the flow of lines and to interprete the colors in order to see the inherent pictures. Such a perception yields a transfer from nature to art.

The Nankai Institute in Tianjin devoted the academic year 1993/1994 to Representation Theory and it was a pleasure for me to stay there for several weeks. For the same period, also Auslander and Reiten had been invited, and later Puig arrived. The discussions with all the participants were very fruitful. I presented the idea of adding infinitedimensional indecomposables to the Auslander-Reiten quiver (which is made from the finitedimensional indecomposables) in order to sew together different components. What one obtains in this way may be called an Auslander-Reiten quilt.

I was happy to have again a bicycle at my disposal and I used much of the free time to cycle around. Once Auslander was asking me where I had been. I told him of all the temples I had visited, the old traditional houses I had seen and so on. He was amazed to hear about temples and pagodas—— he had asked Guo about the city and was told there would not be any attraction. So when he spoke to Guo again, Guo replied: There is nothing special, it is just like any other Chinese city. What seems to be the usual thing for the local people may be of great interest, even surprising, for foreigners: I remember, when I went with Professor Liu to one of the cellars in Antwerp which serve a big variety of beer, with wooden tables and candle light (Antwerp is very proud of these places), he told me that it is very difficult for him to understand why it should be nice to have candle light, when one could have a proper electric illumination!

During the visit to Guilin, the Humboldt applications for Pu Zhang and Lian-gang Peng were planned, both were successful and strengthened the ties between China and Germany. We also arranged a visit of Jie Xiao to come to the Bielefeld Sonderforschungsbereich (SFB). In addition, Chang-chang Xi (after finishing his Ph.D.) continued to stay at Bielefeld for some time (also as a member of the SFB). 1995 saw the start of the official cooperation between BNU and the Faculty of Mathematics at Bielefeld University. And there was an Oberwolfach meeting at the end of July 1995 where Chang-chang Xi, Jie Xiao and Pu Zhang were able to participate.

The Volkswagen Cooperation

During Xi's time at Bielefeld he had cooperated with Steffen Konig. In order to continue the joint investigations, Konig contacted the Volkswagen Foundation trying to find out whether they would provide travel support as well as means for computer equipment and literature. The Volkswagen Foundation had supported scientific contacts between Germany and China before, however this program was suspended after 1989. But it was clear that they had the intention to start again, with projects on a broader scale. They asked us to outline such a proposal, including several research groups in China as well as in Germany, aiming not only at the exchange of scientists and the improvement of the working conditions in China, but allowing also Chinese Ph.D. students to come for a year to Germany. We got very helpful comments on our first draft, with an indication to include further items, for example means for German language courses (and the provision that the final application should not comprise more than the double amount of the first draft). The project was coordinated on the Chinese side by Professor Liu, on the German side by myself, it included universities in Beijing, Changsha, Chengdu, Hefei, Xiamen as well as the Academia Sinica, and the German universities at Bielefeld, Chemnitz, Dusseldorf and Paderborn.

Let me mention the eight topics which were outlined in the application:

· Structure of the module category of a finitedimensional algebra, combinatorial invariants, generic modules.

· Homological investigations, tilting theory, functor categories, derived categories.

· Vectorspace categories, matrix problems, reduction algorithms.

· CREP (computer algebra programs for dealing with combinatorial problems in representation theory).

· Hall algebras and quantum groups, canonical bases.

· Cellular algebras, quasi-hereditary algebras.

· Cohomology of Schur algebras and of symmetric groups, Hecke algebras.

· Kazhdan-Lusztig theory.

This list shows the wide variety of joint research themes; not all the open problems mentioned in the application had been solved by the end of the program, but the final report documented essential advances for nearly all of the topic, and in some cases the results obtained exceeded by far the envisioned aims.

For the preparation of the Volkswagen project I traveled to China in 1997 and visited Beijing, Hefei, Xi'an, Chengdu, Xiamen. After this visit, I felt it would be necessary to learn both to speak and to write Chinese. However I did not succeed at all.

The project allowed five Ph.D. students and one postdoc to come to Germany and actually all of them stayed most of the time at Bielefeld. The first two, Yang Han and Bin Zhu actually obtained their Ph.D. there (this was not the intention of the project, but it turned out that the essential part of the research work leading to their Ph.D. theies was achieved during their time at Bielefeld, so we arranged that their stay could be slightly extended).

The exchange program comprised of mutual visits (up to two months): candidates from China were Jie Xiao, Chang-chang Xi, Jin-yun Guo, Liangang Peng, Pu Zhang, Ying-bo Zhang, Bang-ming Deng and Ya-nan Lin.

Those from Germany were Draxler, Kerner, Konig, Lenzing, Happel and Unger. In addition, the Volkswagen Foundation provided means to buy some computer equipment for Sichuan and Xiamen University and books for BNU and Hefei.

The project also included the support of a scientific conference to be held at Beijing; the planning of this conference was done quite early and it turned out that it was possible to organize the conference (ICRA 9) in 2000 in the frame of the ICRA-conferences, thus as a truly international conference. This will be discussed later.

A kind of intermediate ICRA-conference (ICRA 8.5) was held at Bielefeld in 1998. Through the Volkswagen fund, but also many other supporting measures from China as well as from Germany, fourteen Chinese mathematicians were able to participate (Chang-chang Xi, Jie Xiao, Yang Han, Bin Zhu, Bang-ming Deng, Lian-gang Peng, Shao-xue Liu, Ya-nan Lin, Xian-neng Du, Jin-yun Guo, Hai-lou Yao, Pu Zhang, Shun-hua Zhang and Ying-bo Zhang). All of them reported on recent investigations and Jie Xiao was asked to give a plenary lecture. We were very happy that Professor Liu himself was able to come. This was his second visit to Bielefeld. For that travel, as well as a third one in 2000, his wife could accompany him.

As a by-program to the conference, an excursion to the old city of Lubeck was arranged, with a concert devoted to Mahler's second symphony (it is rather rare that this symphony is played since it requires a lot of resources). After the concert, Liu started a discussion on the meaning of the tunes, on the images delivered by the various sounds and phrases. It was important for him that art should have a meaning, that it is necessary to understand the internal structure as well as the hidden pictures.

When Professor Liu retired there was a great danger that his integration power would be lost. The representation theory seminar of BNU split off in several distinct seminars, with only sporadic joint sessions. Of course, such developments are difficult to control (and a similar development oc-

curred in Bielefeld soon after). On the other hand, by that time his efforts to built a large representation theory group in all of China were bearing fruits and the topic was no longer restrained to Beijing. Now, obviously there is a very fertile cooperation between the different groups across the country. A first survey on the development of representation theory of finite-dimensional algebras in China was written by him and Pu Zhang already in 1996 and has appeared in a volume on Rings, Groups and Algebras in China published by Marcel Dekker, which is also included in this volume.

ICRA 9

ICRA stands for International Conference on Representations of Algebras, these conferences had been held before in Canada and in Mexico, in Japan and in Norway, the first one was organized by V.Dlab in 1974 at Carleton University in Ottawa, Canada. ICRA 9 was tentatively scheduled to take place in Torun (Poland), however during the Norway conference in 1996 it became clear that there also would be the possibility to hold the conference in China. At that time, it was not yet specified whether this should mean Beijing, or Hefei and Chengdu: usually, ICRAs have been accompanied by a corresponding workshop, and sometimes conference and workshop were held at different places. The scientific committee unanimously adopted the proposal to hold the 2000 conference in China (leaving open the precise location) and Professor Liu was asked to join the committee.

I visited Beijing in 1999, in the frame of the Volkswagen project. During this visit, we discussed the organization of the ICRA conference as being part of the cooperation project. I always felt that one should try to hold the opening of such a conference in the old Confucian academy: it still exists, being surrounded by the buildings of the national library, near to the Confucian temple in the north-east of Beijing. The Chinese colleagues tried to get the permission but apparently failed. I also joined the local organizers when they approached the Chinese Research Council for obtaining financial support. (At that time, we were informed that a Chinese-German science center was going to be established, supported by the Chinese Academy of Sci-

ence and the German Research Council. The main purpose of this science center is to host binational conferences, and one may hope that it is possible to organize such a conference on Representation Theory of Algebras in a not too far future.)

During the workshop of ICRA 9, the Chinese representation theory group gave two series of lectures presenting their own contribution to the subject. The lectures by Lian-gang Peng and Pu Zhang outlined the progress on Hall algebras, the title was Twisted Hopf algebras, Ringel-Hall algebras and IM-Lie algebras, whereas Bang-ming Deng and Chang-chang Xi spoke on Quasi-hereditary algebras and Δ-good modules. On the other hand, on the request of Liu I gave a survey with the title Combinatorial Representation Theory — History and Future, including a list of open problems and directions for future research.

The focus of conference and workshop was on questions in the representation theory itself, to a lesser part on possible applications. I felt a little unhappy that the interaction with group theory and with topology, but also with some parts of Lie theory remained undiscussed, since some of the invited main speakers who were supposed to outline such relations were not able to come. In addition, a conference with a similar target was held in Rumania just some weeks before and may have distracted several of the possible participants. Despite all these obstacles the conference was clearly a great success. It provided a vivid picture of the subject and stimulated further progress. The two volume proceedings of workshop and conference (edited by Ying-bo Zhang and Dieter Happel, and published by BNU press) record the main developments.

At the time of the conference, another convention took place at BNU: a meeting in didactic of mathematics, concerned with the teaching of mathematics in China. It was discussed whether (or not) China should follow more closely the teaching concepts developed in the West. This seems to be a quite curious proposal, since all the comparative studies show the superiority of the traditional teaching used in the East. But it requires that students

work hard, without being allowed to blame the environment of the teacher for any failure. Of course, learning by heart alone is not sufficient for a proper understanding, but it is a necessary requirement for building up longer range considerations. And as intensive training in sports, the final success should provide the student with satisfaction.

When I visited Dunhuang, a city in the middle of the desert, I was amazed to see in a bookstore a large number of booklets with mathematical exercises. They looked like school books, but I learned that they are bought to be used by the children after school, and to allow the parents to control what the children have learned. A lot of these booklets were edited by an institute at BNU devoted to the problems of teaching mathematics. I should note that Bielefeld is famous for having a correspondingly named institute (the IDM), but its impetus is of a completely different nature: one of its endeavor was a proposal to reduce teaching of mathematics for those students who do not aim towards mathematically oriented professions: why should they be confronted with quadratic equations, with sine and cosine, or with mathematical proofs? It is a pity that mathematics often is considered as a burden and not as a source for inspiration and understanding.

For quite a while Professor Liu has now been engaged in mathematical education both at university, as well as high-school level. I remember that already in 1996 he complained to me that students who have studied undergraduate algebra will know the van der Waerden approach, but will not be aware that groups were actually designed in order to describe symmetries. He took a lot of effort in order to remedy this situation!

2002

In 2002, the International Congress of Mathematics took place at Beijing. Every four years, such a congress is organized somewhere in the world, and it is considered as the main gathering of mathematicians. During the congress I visited BNU twice. There was a tour around the campus for some of the participants of the IDM, and during the final dinner we praised the official cooperation between BNU and Bielefeld University which has

been extended to include analysis and stochastic as well. And I was invited to give a lecture on recent developments and possible future directions in representation theory. I choose to report on ICRA 10 which had been held some weeks before at the Fields Institute in Toronto (Canada), since unfortunately not many Chinese mathematicians had been able to participate in this conference.

The present state: The Chinese group in representation theory is now working on a big variety of topics. Let me give just a few indications: Ying-bo Zhang devotes her time to the very delicate representation theory of BOCSes. Chang-chang Xi deals with quasi-hereditary algebras, the representation dimension, the finitistic dimension conjecture, with Hecke and Brauer algebras. There is a large group working on the Hall algebra approach to quantum groups, namely Xiao from Tsinghua University together with Peng, Deng, Lin and many others. Their contributions are highly appreciated, they are published in leading journals and often quoted and used (and there is some parallel work by Pu Zhang and by Guo). Despite the diversity of the approaches and thoughts, all the groups unite at least once a year for several days, a very important measure for the exchange of ideas.

The Volkswagen project found its continuation by an EU-project under the scheme "Asia-link". The main coordinator is Steffen Konig (now at Leicester), the Chinese coordination is in the hands of Bang-ming Deng. It includes four research groups in China (Chinese Academy of Sciences, Beijing Normal University, University of Science and Technology of China, Hefei and Tsinghua University, Beijing) and five in Europe (Leicester, Leeds, Chemnitz, Bielefeld, UIA Antwerp). It started in October 2002 and will run for three years. As part of this Asia-link project, there will be two conferences held at Beijing in 2005, the first one, in May, will have a broad coverage of topics, the second one is devoted to a more special topic, namely to the theory of triangulated categories.

Moreover, there are plans to establish a joint Chinese-German graduate school. An application (by Henning Krause and Bang-ming Deng) is sched-

uled to be handed in by the end of 2004.

Looking at the ongoing and indeed increasing cooperation, one should note the change which has occurred: Whereas in the early years the flow of ideas was mainly in one direction, there is now a real partnership, and actually China takes a leading role in several research topics. In this way, the vision of Professor Liu expressed in 1985 has been very much fulfilled.

Being introduced by him to the Chinese vision of science and art, this was very valuable for me: I learned from him a deeper understanding of culture. He introduced me to Chinese culture, but at the same time I obtained also completely new views on the European one. His attitude towards arts and culture is the same as the one he exercises towards mathematics: First of all, there is his obvious ambition for perfect presentation. But of course, the main concern has to be the content: the concentration on basic and important problems and the vision of a unified global theory. The struggle between general assertions and concrete examples cannot be resolved in a one-sided way, both directions are of importance. And finally, his drive for understanding the internal structure, for elaborating hidden images corresponds to the old tradition to visualize algebraic results using combinatorial devices. The representation theory of finitedimensional algebras has a very strong combinatorial flavor, so it does not seem to be surprising that he was attracted by it very early.

Bielefeld, 30, 8, 2004　　　　Claus Michael Ringel

自　序

Preface by the Author

我于1929年11月6日出生在辽宁辽阳．“九一八”事变后随父母到北京，就一直生活在北京．父亲刘荫厚是赋闲军人，母亲吴洁是家庭妇女．我于1959年结婚，妻曹文龄，中学物理教师．我们有两个可爱的女儿：刘艺冰是北京师范大学毕业，学数学；刘路是北京大学毕业，学生物化学，后在比利时取得博士学位．

我从小在许多方面都表现平常，只是念书、学数学还行．我从五中初中毕业（1942）后，考入北平高级工业职业学校，那里只有理工课程而无文科课程．我选择北平高级工业职业学校的原因是：在那里能学得一技之长，就可以工作养家了．许多数学老师给我留下很深的印象，特别是五中的大蔡老师和高工的李欧老师．大蔡老师把平面几何讲活了，他对数学玩味的那种学者风度潜移默化着学生．李欧老师朴素、潇洒、风度翩翩，听他的大代数课是一种美的享受．我之所以当时立下志愿做一名中学数学教师，除了怕复杂

* 本文是刘绍学自传．写于1999年．用之以代序．

的处世哲学和喜欢纯真的学校生活外，是数学老师们，特别是蔡、李老师吸引我走上这条路．新中国成立后李欧老师成为清华大学数学系的著名教授．在20世纪80年代末我登门拜访他，两位白发师生终于在将近50年后再次见面．

想上大学的强烈愿望，使我在高工土木系两年（1944—1946）之后转到辅仁附中高三，在那里念了两个月（我的数学老师是朱鼎勋和孙梅生）后，于1946年11月考入北平师范学院（今北京师范大学）数学系．我家境贫寒，北平师院给学生生活费．我表兄于逢原，当时是一位很有数学天分的数学系学生，他对我说："上数学系全靠自己学．"

我的大学生活是安静、单调、平平庸庸的．那时数学系课程很少，如线性代数、微分几何、概率统计、偏微分方程、泛函分析等，我在大学期间都没有学过．然而听过很多名家的课，如傅种孙、张禾瑞、段学复、王湘浩、闵嗣鹤、赵访熊、胡世华、秦元勋等．对于我来说，听名家的课比学习数学教学方法课获益更多．特别是傅种孙先生的那种深挖教材，讲体会、讲联系的教学风格深深地影响着我，我一生都在努力模仿他．

自学是大学生活中的重要部分．至今仍能记得当年自学高木贞治的《解析概论》，弄懂了隐函数存在定理的证明时，自己的那种得意忘形的神情．更难忘的是，在1948年的炎热暑假中，苦读硬译了吉田洋一的《实变数函数论》，我把译稿拿给闵嗣鹤老师（他当时教我们实变函数论）看，他细声细语，想说又好像不好意思地说出的那两句表扬我的话．老师对学生说在心窝上的表扬是非常有分量的．我又记得，在莫斯科大学学习时，有一次向导师 A.G.Kurosh 汇报自己的论文工作时，他对我说："您要像这样乱用超限归纳法的话，您大概会给我带来很多'漂亮'而'杰出'的定理来．"现在想起当时的尴尬场面，仍有无地自容之感．然而这却是使我受益终身的"骂声"．

我第一次听到群的定义是师兄王世强在1948年全系跨年级的学生讨论班上作的报告中．他一上台就在黑板上写下群的四条公理，然后就是一些天书般的词句．虽然我当时没弄懂什么是群，然而自学加上讨论班，使我对数学学习很自信了．我再也不怕数学了．我开始"怕"数学是1982年我已做了两年数学教授之后的事了

我于1950年在北师大毕业后便留校工作一直到现在．新中国的成立和傅种孙先生的厚爱，使我有机会于1953年9月去莫斯科大学学习．

我的留苏生活（1953—1956）仍然是安静、单调、平平庸庸的学习生活．这实际上也是我这一生的生活模式，这该是我“少无凌云志，但求闲散心”的生活态度的自然结果．

在国内留苏预备部中，我申请的学习方向是实变函数论．至今我也不清楚如何改成为近世代数的．我乐于接受这一改动，甚至今天想起来还有点后怕：若是我真的在苏联学起函数论来，不可想像现在的我该是一个什么样子．出国之前我曾在傅种孙先生那里第一次学习近世代数课，用的是G.Birkhoff和S.Mac Lane的《A Survey of Modern Algebra》，后来又听过张禾瑞在北师大讲体论．1950年起和袁兆鼎一起去北大听张禾瑞先生的代数的结构课以及参加他主持的代数讨论班，在那里认识了谢邦杰和张芷芬．1951年，张远达、袁兆鼎和我在北师大组织讨论班读E.Artin的小书《Rings with Minimum Conditions》．我是喜爱代数的，当时只是觉得函数更接近人间烟火，而群、环、域太远离尘世，为了更好地报效祖国，所以才选报了函数论方向．所幸没能实现．

在导师A.G.Kurosh指导下的三年（1953—1956）研究生生活中，有三件事印象深刻．第一次见导师时，就指定他1953年刚出版的书《群论》叫我读．这是一本厚厚的、总结从有限群论向无限群论发展的著名的书．在之后的两个月里，我从头至尾地把它读下来了，对于数学中的“推广”以及代数中对“结构”的研究有了感受．其次是，当确定我将在环论方面做论文时，我几乎细读或粗读了20世纪40年代以来的关于环的结构方面的所有论文，这些正是N.Jacobson在其1956年出版的《Structure of Rings》一书总结的成果．去莫斯科前我最怕的就是做论文．然而读了这一批文章之后，我的感觉已是：熟读唐诗三百首，不会作诗也会吟了．这种感觉当然是“好极了”．第三件事是梦中得解．我曾猜想，局部有限代数借助局部有限代数的扩张仍是局部有限的，这命题对Jordan代数也该是成立的，因为对交错代数情形我已证明，对Lie代数情形虽然这命题确不成立，但我也已给出它成立的充要条件．但长时间苦思冥想这样一个小的具体问题，却始终不得其解．然而一天夜里梦得一想法，喜醒时记下，第二天上午终得一证明．这是我一生中

仅有的一次.

Kurosh 有他自己的科研想法，然而我从他那里学到的科研之道是：手中掌握一些研究对象（如结合环、Jordan 环、Lie 环、群等），还掌握一批结构定理，在学习一些新的结构定理时，对它们作各种推广或平移到其他一些研究对象上．我一生就是在这个框架下写数学论文的：一方面不愁没有问题考虑，另一方面自己也深知这样做是得不到有深度结果的.

1956 年获苏联副博士学位后，回国在北师大数学系继续工作．我的数学生活和我的同龄数学工作者该是类似的．具体说我的时间表如下：1956 年学习代数（环论）后回国，1958 年就放下代数去搞实际问题，和学生们一起摇计算器，计算刘家峡水坝的应力．1959 年回到代数，协助张禾瑞先生办了一个代数研究班．半年后，班散课停，再搞实际问题，和学生们一起再摇计算器，计算北京电视塔的振动频率．1960 年改搞计算数学．1962 年北京龙王庙会议后又回到代数．1964 年我招第一个代数研究生漆芝南，并一起下乡搞四清．1965 年和研究生一起回校念 Lie 代数．1966 年至 1976 年是文化大革命．1979 年开始招代数硕士研究生．1979 年成为教授．1981 年成为博士生导师，1982 年起招环论方向博士生．1999 年我的最后一个博士生朱彬毕业．我也就结束了我的数学教学生活.

1982 年开始指导第一位博士生罗运纶，使我在教书生涯中第一次感到不能胜任愉快了．这得从我的科研领域谈起．我搞代数三起三落，再加上我的闲散和满足，使得在 1978 年起又重新搞代数时基本上还是从 1956 年我当时的那个水平出发．环的经典结构理论的基本框架就是 Wedderburn 结构理论及其各式各样的推广．我的副博士论文就是对结合代数、Lie 代数、Jordan 代数以及交错代数的 Wedderburn-Malcev 型定理的推广，因而我对它是熟悉的．然而我对 1958 年出现的 Goldie 定理、Morita 对偶、等价理论以及后来的环（模）论中的同调方法就不熟悉了．这些在我 1983 年在科学出版社出版的《环与代数》一书中可以清楚地看到．借助在环的结构理论中搞过一些科研的经历，如果说在教本科课和指导硕士生时，还有点胜任愉快感，在指导博士生时我就力不从心了．我深知科研领域对博士生的重要性，我深知我熟悉的科研领域

既纯又窄，像在沙漠中流淌的小河，对有漫长前程的年轻人不是一个好的科研方向. 为学生选择和我不太远又是好的领域，这使我真的怕数学了：数学是简单而确切的，弄懂它不太困难，但数学的单纯和精确使它比其他任何学科都走得更深更远，因而理解它，特别是能在数学世界中具有洞察力和想像力，对我是高不可攀的事，选择领域谈何容易. 在学习一些 PI－代数和 Torsion Theory，仍觉不合适而放弃后，我接触了一点代数表示论，又想起段学复先生的话（大意）："只搞结构不搞表示，不够全面." 在没有太多可选择的情况下，便贸然选定代数表示论. 当时我对此领域的前途是没有把握的，但搞懂它是有信心的. 学习曹锡华先生在华东师大取得的成功经验，我和四位刚入学的博士生一起于 1985 年组织代数表示讨论班，苦读这方面的基本文献，同时又请来外国名家，如 M.Auslander（美）、C.M.Ringel（德）、I.Reiten（挪威）、V.Dlab（加）等在我们的讨论班作系列报告. 近 15 年的坚持和努力，中国代数表示论小组终于得到国内代数界和国际代数表示论界的肯定，站住了脚根，还在 1991 年得到北京市高校优秀教学成果奖一等奖. 当然我心里很明白，没有学生们的顽强努力，没有国际表示论俱乐部，特别是 Ringel 教授的帮助和友情，我们的工作是做不好的. 在代数表示论之后，我还曾提倡过微分算子环和 Gröbner 基理论，应该说这也是从环论能够转过去的好领域. 然而由于可理解的一些原因，这些方向没有在北师大站住脚跟.

科研是我教学生活的一个有机组成部分. 如果有人问：你搞的那些没有一点应用价值，也没有什么科学意义的科研，有什么用啊！我会理直气壮地回答：对我的教学非常有好处，我的教学是非常离不开我的科研的. 我这里指的不仅是研究生的教学，更是指我给大学生的教学. 我几乎教过本科中的从解析几何、近世代数到偏微分方程的所有课程，我在教学上对我自己的要求是模仿傅种孙那样的讲体会、讲思路、讲来龙去脉. 数学中的那些美妙的和谐、神奇的联系，常使人感到这不是人做出来的数学，而是一种"神学". 应该承认，在课堂上有许多时候我是把数学作为"神学"硬着头皮搬给同学们的，对此我心中是有歉意的. 多亏我的科研经历，它在很多情形下帮助我理解或设计出书上的定理是如何在人们的手中试验、摸索和制作出来的. 当学生们听着这样的讲述

而面露会心微笑时，我最认识到自己科研的价值和意义.

我对自己的科研是有自知之明的：当一个人对数学世界的认识就像是只摸到大象耳朵的一小部分的话，那么他很难理解大象，更不用说帮助大象了. 但20世纪50年代初，北师大的科研气氛几乎是零度，因而1950年前后大学毕业而在北师大工作的人都认识到，要把在北师大创造一个好的科研气氛作为自己责无旁贷的责任. 为了证明自己的“存在”，也为了创建科研气氛这些“世俗”目的，在我时断时续的科研生活中曾硬着头皮强行做出和发表一些我自己也不太喜欢的文章. 然而另外一些文章，它们虽是生长在数学世界边远地区荒芜园地上的一些小草，由于是我亲手栽培的，我对它们是喜爱的和有感情的. 人是要保持一点孤芳自赏的情趣的，否则生活会变得太缺少花色了. 这种情趣鼓励我写出下面这些结果来.

在副博士论文［3］（指论文目录中的3，下同）中，如前面已经说过，我证明了局部有限代数借助于局部有限代数扩张而得的Jordan代数仍是局部有限的. 在此基础上，K.A.Zhevlakov和我独立地证明了Jordan代数（环）的Levitzki根的存在性，这构成了由K.A.Zhevlakov、A.M.Slin’ko、I.P.Shestakov和A.I.Shirshov这些Kurosh的学生们写的书：Rings that are nearly associative（俄文书，1978，英译本，1982）中第四章的主要结果. 在［3］中我还证明了，局部有限代数借助局部有限代数扩张而得的Lie代数，如果它还是代数的Lie代数，则它必也是局部有限的. 由此结果立刻可知：可解Lie代数是局部有限. 但我对此毫无感觉，而我的师兄也是我的论文的审查人A.I.Shirshov却对我说：“这个结果可视为Lie代数中Burnside题型中的一个有趣结果，而你毫无反应，这一点在你的答辩会上我是要骂一骂的.” 1981年我在芝加哥访问时看到K.A.Zhevlakov和I.P.Shestakov的一篇文章On local finiteness in the sense of Shirshov，Alg. and Logic 12:1，(1973)，其中在一页上引用我的论文［3］5次，都是涉及上述两结果的. 我当时的感觉是：在中苏关系很不好的情况下，这样多的让刘绍学的名字在自己的文章中出现，这只是为了向Kurosh的唯一中国学生表示一下友好的感情. 1989年在新西伯利亚和这些师侄们共同洒泪Shirshov墓前，是这种友情的又一次宣泄.

一个结合代数，如果它的每一个子代数都是理想，就叫做 Hamilton 代数. 我在［12］中给出了这种代数的完全刻画. 美国人 D.L.Outcalt 把它推广到幂结合代数情形，而 R.L.Kruse、D.T.Price 的专著 Nilpotent Rings（1969）把我的这个结果收入到它的第九章中. 我喜欢它是因为它虽然简单，却是我第一个非推广非平移的结果. 一步步摸索前进，最终得到结果的过程是使人非常愉快的.

我们给出了有向图的路代数的同构定理［25］和赋值图的张量代数的同构定理［38］. 这样，例如关于有向图的几何研究就可归结为关于其路代数的代数研究. P.A.Grillet 在 Isomorphisms of stratified semigroup algebras, Comm. in Alg. Vol. 22（1944）4417～4493 一文中指出我们上述结果是仅有的两个关于半群代数的同构定理后，开始了他对半群代数的同构定理的系统研究.

我们在［31］中把群 G－分次环 A 的 Smash 积 A＃G 的概念推广到 G 是无限群的情形，并得到相应的对偶定理和上对偶定理. M.Beattie 做了同样的事，证明方法不一样. 她的文章是 A generalization of the smash product of a graded ring, J.Pure Appl. Alg. 52（1988）216～226. 1988 年后国际上出现很多讨论 A＃G 而 G 是无限群的情形，并多引用 Beattie 的文章. 实际上这两篇文章的主要结果是一样的.

我和 Beattie 等给出了分次本原环的结构定理，其证法是平移 Jacobson 关于本原环的证明，再加上两个小引理，但结果是非常完整而漂亮的. 我在一些场合报告几次，大家都很喜欢分次本原环的这个简明自然的刻画. 之后，C.Nastasescu、J.L.Gomez Pardo 在 Topological aspect of graded rings, Comm. in Alg. 21（1993）4481～4493 中引之为例，对有限分次拓扑做了进一步的一般讨论.

我的最后一篇文章是 1996 年在巴西圣保罗大学访问和 F.U.Coelho 合作写出的 Generalized path algebras，在西班牙的 Murcia 城召开的强调环论与代数表示论联系的国际代数会的会议录上刊出. 见［57］.

我在国内外的数学杂志上发表了 50 余篇论文，获得过 1988 年度的原国家教委科技进步二等奖，获奖项目是“环的结构与表示理论”.

“数学家”是一个美丽的称号，虽然它没有一个清楚的定义，但许多人都有一个自己的理解. 我心目中对数学家有一个非形而上学的定

义，我说不清楚，但我可以说："王世强是数学家，刘绍学是一个合格的教授，是一个好的数学教师，但不是我心目中的数学家." 1999 年在北京郊区龙庆峡那高山秀水的大自然美景中，我向袁向东谈起这个想法时，他是很感兴趣的.

无论在校内外还是国内外，我始终生活在和谐愉快的代数俱乐部中. 和数学名家、代数同行的交往，是我数学生活以及友谊生活中非常美丽的组成部分. 这些是我经常重温而常重现在我面前的宝贵镜头.

在 1950 年刚留校工作时，傅种孙先生对我说："助教是一个过渡的岗位，上不去的人就应该走开." 1955 年在莫斯科，Kurosh 对我说："不能还是像大学生那样按功课表学习，您是研究生了，您的主动学习在哪里?" 我从未见张禾瑞先生在任何场合炫耀自己非常出色的博士论文，而段学复先生对我说过："我钦佩张禾瑞做学问的实实在在." 张禾瑞和 Kurosh 两位老师对周围的人友好和善，从他们身上我感受到教书育人的力量. 师兄王世强的话："凡事要经得住历史的考验." 这有时使我愧对人生，但却令我终身受益. 我一生搞环论，也始终带着问题：环论有什么用? 我不敢去问 Kurosh，从在讨论班中我对他的理解，他一定会是带着不屑的神情回答我：数学中不能谈问题的有用无用，只能谈这些问题有没有水平，修养高不高. 在莫斯科做论文最困难的时候，我和师兄 A.I.Shirshov（后来的院士）谈及这个问题，他说："对数学问题最需要的是，不搞出它来，你就觉得心里不舒服、难受、睡不着觉." 也许他这话对我克服论文中的难关起过一点作用. 1985 年在奥地利的 Krems 参加根论会议，参观一个教堂时，我问与我同行的一位根论名家这个问题，他开始不知所措，之后便说："在我搞根论买了汽车又买了房子后，就再没有人向我提出这样的问题了." 后来偶尔有人问我这一问题时，我总是客观地转述上面意见而没有任何创造性的回答，只是反复强调：搞环论研究无论如何对教学质量非常有好处. 与此有关的是在 20 世纪 90 年代初，有一次在打完乒乓球散步时，曹锡华先生对我说："对范畴定义和讨论 Jacobson 根（指我发表的文章 [33]），有什么意思啊?" 当时我还解释了几句，但事后一想，这除了证明我的"存在"外，也真的说不出来有什么意义.

1962 年在颐和园龙王庙会议期间，万哲先学兄对我说："搞典型群

研究我们掌握一些基本手法和招数，你们搞环论的有哪些基本手法和招数?”问题提得很好．大概因为我不明确（也许根本不存在）搞环论有什么手法，当然也就回答不上来．我在会议上介绍了范畴论后，万学兄对我说：“对这样的东西（指范畴），我感到不知如何去搞它．”我当时却感到，对它还是可以下手去搞的．20世纪80年代，一次出数学竞赛题，我与华罗庚先生在他房间闲谈，他对我说：“国外把我说（骂）成是玩矩阵的魔鬼……表面上你看我搞的是多复变函数、偏微分方程，实际上骨子里还是我的矩阵技巧．”联想到万哲先学兄的话以及1955年华先生在莫大研究生宿舍和我们聊天，当知道我是来苏学习代数时，他非常果断地说：“学代数不必出来，可以在国内学．”我清楚地看到，不同导师、不同领域、不同学术观点对青年学生的影响是巨大的．也许先具体再抽象，先深入再宽厚，更适合年轻人的发展．与华、万的交谈，对我20世纪80年代转换领域是有潜在影响的．

除了做了将近两年（1959—1961）的计算数学教研室主任外，我一直在北师大代数教研室．北师大的这方代数沃土是由傅种孙先生开始，自1952年起在张禾瑞先生的主持下，以后由王世强、郝钢新、吴品三以及我等协助，经长期努力后形成的．张先生的书《近世代数基础》，以及与郝先生合写的书《高等代数》，奠定了北师大以及高校代数教学的基础．张先生在20世纪50年代主持的几个代数研究班，还有进修班和本科，培养了许多代数人才，如湖南师大的李传和、陕西师大的雷天德、福建师大的陈昭木、哈尔滨师大的周汝奇、张之凰、广西师大的程福长、河北师大的朱元森、山东聊城师院的杨子胥、上海师大的孔宗文、席德茗、北京师大的蒋滋梅等等．他们在“文革”前是全国师范院校的教学骨干，“文革”后是培养环论方面硕士生的积极力量，培养出许多优秀的学生．我是又高兴，又带有几丝惭愧地享受着张先生营造的这个代数大家庭的温暖．张先生的这些学生以及其他代数同行组织了我在各地（哈尔滨、西宁、乌鲁木齐、屯溪、杭州、西安、桂林、大连、烟台等）讲环与代数、群论等，有时是和吴品三、许永华、谢邦杰、曹锡华、丁石孙、冯克勤等中的一些人一起讲．从20世纪80年代初开始每两年一次，我和吴品三师兄主持了顺序在江西师大（陈培慈）、扬州师院（方洪锦）、云南大学（王俊民）以及陕西师大（雷天德）召开的

根论或根论环论学术会议. 在最后一次会上, 我是带着一种误导青年进入根论的歉意说出了如下意见: "在既纯又狭、前途不太光明的根论中取得一些科研经验是好的, 但不要在此领域久留, 特别是年轻人要及时开拓出路." 1991 年秋在程福长教授的大力支持下, 我和 H. Tachikawa 在桂林广西师大主持了第一次中日环论国际会议, C.M. Ringel (德)、B.J. Müller (加)、M. Beattie (加)、李白飞、方园、万哲先等也参加了会议. 1992 年在日本非正式地出版了会议录. 1994 年夏在朱元森教授的大力支持下, 我和朱在石家庄河北师大主持了国际根论与环论会议, 来自 11 个国家的五十多人参加了会议. 1996 年出版了这个会议的会议录 [著作, 8]. 第二届日中环论会已于 1995 年在日本开过, 而第三届已变成韩中日环论会议而已于 1999 年 5 月在韩国召开. 1999 年 10 月全国代数学术会议在北师大召开, 北大的张继平和北师大的张英伯两位教授紧张筹备的. 这次代数会喜逢周伯勋、曹锡华两位先生 80 寿诞, 又正好在我 70 岁生日的前夕, 届时会议参加者也带来各地代数朋友们对我们生日的祝贺. 生活在这样一个和谐友好的代数大家庭中, 谁能不感到愉快和幸福呢!

20 世纪 90 年代初, 当张禾瑞先生培养的代数大集体由于成员的老化退休已渐衰落后, 由我的学生们汇成的代数小家慢慢形成起来. 我的 19 名博士生中有四人经联合培养在国外取得博士学位: 张英伯、林亚南、朱彬在德国, 导师是 Ringel, 邓邦明在瑞士, 导师是 P. Gabriel. 15 名在国内取得博士学位, 其中郭晋云、彭联刚、章璞以及邓邦明还在 Ringel 教授的指导下获洪堡基金, 肖杰获 1997 年国家教委 "跨世纪优秀人才" 基金, 1998 年国家杰出青年基金. 北师大的张英伯、肖杰以及川大的彭联刚* 和中国科技大的章璞已成为博士生导师, 这使得全国有环论博士方向的学校从 4 个 (南京大学、吉林大学、复旦大学和北师大) 增加到 6 个, 不久郭晋云 (湖南师大) 和林亚南 (厦门大学) 也成了博导, 届时这样的学校将会多起来. 1997 年川大组织第一次代数表示论讨论班, 2000 年湖南师大将办第二届, 20 世纪 80 年代初张先生的学生组织讲学活动时, 我是站着讲, 20 世纪 90 年代末我的学生组织

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* 彭联刚于 2003 年获国家杰出青年基金.

学术交流时，我已是坐着听了．这就是历史．

我还享受着国际环论大家庭的温暖．我们的一生没有追星族人的情感，只是向往着世人不感兴趣的数学家：能够和自己念过的书或读过的文章的作者见见面、聊聊天、交往交往，那是非常愉快和有益的事，似乎还能帮助你理解数学．在国内我已接触过许多名家，特别是能和20世纪50年代初代数四大名家——华罗庚、段学复、王湘浩和张禾瑞都有过面对面的亲切交谈．在苏联留学期间我接触了差不多所有的国内翻译出版的苏联教科书的作者．П.С.Алексадров在1956年全苏数学会的一个报告中讲道："数学中有许多美的东西，美是不会失落的．"至今仍似看到他说话时的坚定神态．20世纪50年代读环论文章时，对N.Jacobson、I.Kaplansky、S.A.Amitsur三位环论大家非常敬仰，改革开放后才有机会和他们见面交谈．和Jacobson教授在北师大、新西伯利亚见过几次．有一个小插曲：一位朋友告诉我，Jacobson在南京大学讲学时，闲逛书店，翻阅到我写的《环与代数》一书，凭借书中的一些外国人名，猜到书中内容，并在课堂上推荐给大家．我在1981年访问芝加哥大学时，每星期都有一小时固定时间和Kaplansky交谈．半年后临别时，我问他：环论似乎不是一个好方向，如果我想选择一个新方向，你有什么建议吗？他的回答简单而明确：你还是搞环论吧！我理解为：50多岁的人了，改方向谈何容易．1989年6月我访问以色列时，在耶路撒冷J.S.Golan的家中见到了Amitsur夫妇，一位身材不高的老人，和我从他的文章得到的印象是很不一样的：我想像中Amitsur该是一个很潇洒的人．

1985年，我对欧洲的访问，对于我和我的学生们进入国际代数俱乐部非常关键．今日回想起来是一个稍纵即逝的难得机遇．我收到两封邀请信，一是比利时教授F.Van Oystaeyen第二次邀我去比访问三个月（这是曾在比留学的师弟李文琦推荐的，Van Oystaeyen想邀一位中国代数学者访问比），一是R.Mlitz邀我参加在奥地利召开的国际根论会议（他是查Math.Review看到许永华和我的名字，向《数学学报》编辑部了解到通讯地址后向我们发出邀请的）．是年初，我突发较严重的心脏病，幸经人民医院胡恒慧大夫的精心治疗，住院三个星期，又征得阜外心内科专家刘力生教授的同意，我才能"毅然"于五月份按期去欧访

问. 这次访问使我和 Van Oystaeyen、Mlitz、C.M.Ringel（当时在德的陈家鼐除安排我对慕尼黑大学 F.Kasch 的访问，还替我和 Ringel 取得了联系，Ringel 邀请我去 Bielefeld 大学访问）建立了联系和友谊. 他们三人顺序作为环论、根论、代数表示论的引路人使我进入相应的国际俱乐部. 之后我多次出国访问，去欧洲、北美、南美、日本、苏联等地，都是与这次访问有关的.

Fred（即 Van Oystaeyen）是一位热情、透明、正义、非常聪明的代数专家，有十多本专著、二百多篇论文，我和他也有合作文章 [31]. 当我第一次看到他和学生们在咖啡馆谈笑中提出数学问题、解决数学问题时，是很感慨的：我们太习惯于在图书馆、在杂志堆中低头想问题了. 我的英语是向 Fred 学的：无论在火车上、汽车上、街道上，他总是把我当作英国人那样不断地、快速地、带着感情地说英语. 这样，我好像也真的听懂了英语. 我对 Fred 谈起，1984 年以来我讲了 Torsion Theory 的课,已编写了四章的讲义. Fred 对我说，Torsion Theory 没有什么前程，不值得写这方面的书. 这促使我放弃了这块鸡肋似的工作. 1994 年在石家庄的国际根论与环论会议上和 J.S.Golan 谈起他的巨著 Torsion Theories, Longman Scientific and Technical, Harlow（1986）时，我说，你的这本书可看作 Torsion Theory 这一分支的结束吧！看来他也同意这种说法. 我和 Fred 共享我俩之间的美好友谊：我对他说，我们是 “two heads with one mind”；他对我说，“I only complain about life to my very best friend (s).”

Claus（即 C.M.Ringel）可以说是上帝在我们转变科研方向的困难时期给我送来的“援助”朋友. 我六次去德，他六次来华，虽相隔万里，但自 1985 年至 1999 年几乎每年都见面. Claus 热情、勤奋、正义、善解人意，对中国有一种特殊的感情. 除上面已提到的，我有三名博士生在他指导下获德国博士学位，有四名学生在他接受下成为洪堡基金获得者外，还经常邀请我的学生们去访问或参加国际会议. 自 1990 年以来每年都有我的两三个学生在他那里. 1997 年—1999 年在“大众汽车”的资助下，在 Claus 和我的主持下有一个德中代数表示论学术交流三年协议，这使得我们的交流更加密切而富有成果. 1998 年 9 月，我和我的 15 名学生居然同时在德国 Bielefeld 大学参加 Claus 主持的一个代数表

示论国际会. 中国代数表示论小组的所有文章都和 Claus 的工作有关, 他所引入的 Hall-Ringel 代数和倡导的拟遗传代数在中国得到充分的发展, 国际代数表示论系列会的第九次会议在 2000 年召开, 又是在 20 世纪末, 且是联合国命名的数学年, 所以大家都对在 2000 年开的这次会很看重. 波兰同行很希望承办这个会. 看得出, Claus 经过多方协调, 冒着得罪波兰同行的危险, 最后在 1996 年在挪威召开的领导小组会上决定委托我在中国组织这个会, 并接受我为该系列会的顾问委员会的成员. 我当时在匈牙利, 事后得知, 我受宠若惊, 没想到中国代数表示论集体和我本人能够享此殊荣*. 从这里我再一次看到, Claus 对中国的情感, 对我们小组的支持, 以及对我本人的友谊.

1987 年在苏黎士, P.Gabriel 陪我去看列宁曾住过的房子, 路上谈起科研时说: “搞科研好办, 自己去做就是了. 学习别人的东西是困难的.” 这和我对科研和念书的感觉刚好是相反的. 记得 1982 年在芝加哥, I.Kaplansky 对我强调的是: “一定在搞科研的同时学习一些与你的科研题目不同的知识.” 这使我理解了, 为什么他的路愈走愈宽. 1994 年, M.Auslander 在挪威去世前, 他和 I.Reiten 和 C.M.Ringel 参加在南开大学举行的中国数学代数年的讲学活动. 他在兴致勃勃地讲完课后, 在去饭厅的路上非常安详自然地对我说: “我有前列腺癌, 没有多少时间了.” 也还是在这次相聚时谈起数学来, 他说: “你相信有限单群的分类定理是已经被证明了吗?” 这突然的问题使我张口结舌: 人们太习惯于少数被尊重的权威说对的事就是对的这样一种规则了. 还是在南开, 我和 Auslander、Reiten 在一起时谈到他俩发现的 AR-叙列时, 他说: “我们最初发现它时, 并没有派上用场, 但对这样一个有特性的叙列, 我们确信它一定是有用场、有力量的.” 现在大家都清楚, AR-叙列是研究模范畴结构的好工具, 是代数表示论的命根子. 大约是 1990 年, S.Montgomery 陪她的丈夫在北大访问时, 在北大的外宾接待室中我们一起回忆了她的老师 I.N.Herstein 之后, 她对我说: “如果再重新开始搞代数的话, 我会选择代数几何的.” 看起来不止一位环论专家都向往着博大精深的代数几何. 看起来, 搞代数的人不把自己的工作和

* 第 9 届国际代数表示论大会 (JCRA) 于 2000 年在北京师范大学召开.

Lie 理论联系起来，或者和代数几何联系起来，是不太会被人看重的. 1993 年，D.Passman 开车把我从 Madison 送到 Iowa 去访问 K.Fuller. 我和 Fuller 谈起环论在美国和中国不太被重视，像 Fuller 这样的环论专家在美国是得不到基金的. 他对我说："我搞环论，就是为了告诉后人，这几个问题已有人解决了，你们可以不必搞了. 至少这个目的是可以达到的." 我是带着同情和怅然的心情听他这样谈论自己的科研的. 1988 年在日本 H.Tachikawa 开车带我去 Tsukuba 山玩，在途中我向他提出召开中日环论学术会议的建议，他欣然同意，并说："日中合作起来是可以和欧洲（在环论方面）抗衡的." 我很赞赏他的这种心态，但同时我也感到惭愧：我想到的只是中日交流而已. 应该承认，在国际交流中，在考虑一些问题中，在我心灵上总有一些穷、弱、殖民地式的阴影，挥之不去. 其实想想，特别是改革开放以来，国家走向富强，我们也已有了一个想吃什么就能吃什么的生活，但在我身上这种劣根性的阴影，总是挥之不去，特别是在西方的学者面前. 1989 年，在新西伯利亚召开的纪念 A.I.Malcev 的国际代数会上，遇到 Rings that are nearly associative 一书的在世的两位作者 A.M.Slin'ko 和 I.P.Shestakov，他们都是 Shirshov 的学生，见到我这位"师叔"分外亲热. 我向 Shestakov 提起他文章中引用我文章的事，他说："那不仅是为了友谊，你的（副博士）论文是作为必读的文章来学习的（该是指有关局部有限性质的那部分，其余都是实质上照抄别人的证明的形式推广之作，不会被要求必学的）. 那时就觉得你很亲近，是自己人." 群众的眼睛是雪亮的，我的被引用的几篇文章都是我花费了心思、自己感到有一点新意的东西. 那些照抄别人证明而得点"新"结果的文章，虽也发表了，甚至是在不错的杂志上发表的，也是无人问津的.

学代数的人要有一点孤芳自赏的情趣，同时也需要同行们的鼓励，否则生活就太缺乏色彩了. 我一生中永远忘不了我的中国、德国、挪威、比利时、美国、俄罗斯、巴西、西班牙、日本、加拿大、匈牙利、墨西哥等国家的代数同行们，以及我的学生们对我的鼓励："一个在环论上已站住脚而在 56 岁的时候和学生们一起转攻代数表示论，是值得敬佩的." "在中国创建一个坚强有力的代数表示论集体，是有意义的，是难得的." 这些都是私下里说的（不是在报奖或提职的推荐信上说

的)，在面对面的个人聊天中说的．很真诚，很美丽，我从中得到巨大的力量与难得的生活乐趣．

最后应该补充的是：我在1979年加入中国共产党．曾任《数学年刊》、《数学进展》、《数学季刊》的编委，《数学进展》的副主编，数学通报的副主编和主编，《高等数学研究》的名誉编委，湖南教育出版社《中学生数学视野丛书》的主编．曾任国家教委高校教材编审委员会数论代数小组的副组长，北师大数学与数学教育研究所所长．还曾在莫斯科大学当过中国留学生体育班长．

还该补充一点的是：我于1996年冬至1997年夏一气呵成地为大学数学系本科写了一本教科书《近世代数基础》．这是我写的唯一一本大学生教科书，它反映着我对这门课的内容和教学的理解，可以说用尽了我的心思．此书作为九五国家重点教材、面向21世纪丛书中的一本，于1999年由高教出版社出版．我现在以一种考生等着发榜的心情期待着读者和教师们的反应．*

* 本书获2002年国家教育部全国普通高等学校优秀教材奖二等奖．

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一 非结合代数

I. Non-Associative Algebras

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北京师范大学学报(自然科学版)
1956,(1):45 ~ 70

无限代数的分解

On Decomposition of Infinite Algebras

§1. 导　　言

在作者的文章[1] 中研究了把关于有限代数分解的定理推广到局部有限代数上去的问题. 本文是[1] 的继续, 并且是建立在[1] 中之结果上. 在这里研究相同的问题, 即把那样的定理推广到无穷(维) 代数(不一定是局部有限的了) 上去. 就作者所知已有四篇文章[2], [3], [12], [13], 讨论过这个问题. 我们将证明对交错代数, 李代数, 若当代数也可得到与[2] 中关于结合代数的一个定理的相似结果(参看 §9), 并且给出[2] 的这一定理中关于分解的唯一性部分一个新的证明. 我们所得到的关于结合代数的定理较[2] 中之结果稍强.

另一方面在[12] 或[3] 中曾给出一个例子, 它说明结合代数 A 不一定是可分解的, 即使当$\dfrac{A}{R}$是有限可离代数的直和而$R^2 = 0$. 我们将给出充分条件(参看 §10) 使这种类型的代数可分解. 并证分解的唯一性.

这文章是在 А. Г. Курош 教授指导下做出的, 利用这个机会向他表示感谢.

* 此文之俄文稿已投给杂志 Математический сборник.

§ 2. 扩张代数的局部有限性

在这一节我们将讨论下面这一问题：由局部有限代数借助于局部有限代数所得的扩张代数是否仍是局部有限的？即，若知代数 A 之理想子代数 R 是局部有限的，且知商代数 $\bar{A}=\frac{A}{R}$ 也是局部有限的，能否断言 A 自己也是局部有限的？

显然，只需讨论当 $\bar{A}=\frac{A}{R}$ 是有限的情形. 设 $\bar{a}_i$, $i=1,2,\cdots,n$ 组成 $\bar{A}$ 之基底. 则有

$$a_ia_j=\sum_k f_{ij}^k a_k+x_{ij},\quad i,j=1,2,\cdots,n. \tag{1}$$

其中 a_i 是陪集 $\bar{a}_i$ 的某一代表. f_{ij}^k 是属于 A 之基本域 Φ，而 $x_{ij}\in R$. 为了证 A 是局部有限的，只需证 a_i, $i=1,2,\cdots,n$ 和任一包含 x_{ij}, $i,j=1,2,\cdots,n$, R 之有限子代数 X 在一起于 A 中生成有限子代数.

令 $x_j(j=1,2,\cdots,m)$ 组成代数 X 之基底. 令 M 是由元素 $a_i(i=1,2,\cdots,n)$ 及元素 $x_j(j=1,2,\cdots,m)$ 组成的集合. 由 M 中之元素可以作任意长度的非结合字，就像在[4] 中所作的那样. 字 b 之长用 $d(b)$ 来表示. 我们也将碰到带系数(属于 Φ 中) 的字的和. 今后用

$$b=b_1+b_2+\cdots+b_t+\backsim$$

表示：在代数 A 中字 b 是字 $b_1,\cdots,b_t$ 及一些长度短于 $b,b_1,\cdots,b_t$ 之字的和. 因之等式

$$b=\backsim$$

也是有意义的. 令 W 是所有至少含一个 x_j 的字的全体. 今后在本节中字母 b 将永远用来只代表 W 中之字(带系数或不带系数).

令 W_k 表 W 中所有长度不大于自然数 K 之字的全体. W_k 是 R 中的有限子集合，故生成 R 之有限子代数 B_k. 若 N 是以 $a_i(i=1,2,\cdots,n)$ 为底的向量空间，则显然 B_k+N 是 A 之有限子空间，且包括所有的 a_i 和 x_j. 要证 a_i,x_j 在 A 中生成有限子代数，只需证存在一自然数 k，使 B_k+N 是代数. 而为了这个，由于(1)，只需证 $W\subseteq B_k$.

现在我们分别对于结合代数，交错代数，李代数和若当代数来证这一事实. 即转到下面四个预理上去.

预理 1　由局部有限代数借助于局部有限代数所得之扩张结合代数是局部有限的.

这个在文章[5]中被证明了,并可由下面较一般的结果推出.

预理 2　由局部有限代数借助于局部有限代数所得之扩张交错代数是局部有限的.

证明:令$[a,b,c]=(ab)c-a(bc)$.我们知在交错环中有

$$[a,b,c]=-[b,a,c]$$

和其他与此相仿的等式.因此[①]

$$\begin{aligned}a_i(a_jb)&=\pm(a_ia_j)b\pm(a_ja_i)b\pm a_j(a_ib)\\&=\pm\Big(\sum_k f_{ij}^k a_k+x_{ij}\Big)b\pm\Big(\sum_k f_{ji}^k a_k+x_{ji}\Big)b\pm a_j(a_ib)\\&=\pm a_j(a_ib)+\backsim.\end{aligned}\tag{2'}$$

同样,

$$(ba_j)a_i=\pm(ba_i)a_j+\backsim.\tag{2''}$$

从 $a_i(ba_j)=\pm(a_ib)a_j\pm(ba_i)a_j\pm b(a_ia_j)$ 得

$$a_i(ba_j)=\pm(a_ib)a_j\pm(ba_i)a_j+\backsim.\tag{3'}$$

同样,

$$(a_jb)a_i=\pm a_j(ba_i)\pm a_j(a_ib)+\backsim.\tag{3''}$$

从恒等式

$$\begin{aligned}a_i[(ba_j)a_i]&=a_i[\pm b(a_ja_i)\pm(a_ia_j)b\pm a_i(a_jb)]\\&=\pm(a_ia_i)(a_jb)+\backsim,\\a_i[(a_jb)a_i]&=a_i[\pm a_j(ba_i)\pm(a_ia_j)b\pm a_i(a_jb)]\\&=\pm a_i[a_j(ba_i)]+\backsim\\&=\pm(a_ia_j)(ba_i)\pm[(ba_i)a_i]a_j\pm(ba_i)(a_ia_j)+\backsim,\end{aligned}$$

得　$$a_i[(ba_j)a_i]=\backsim \quad 和 \quad a_i[(a_jb)a_i]=\backsim.\tag{4'}$$

同样,

$$[a_i(a_jb)]a_i=\backsim \quad 和 \quad [a_i(ba_j)]a_i=\backsim.\tag{4''}$$

令 $k=2n+1$,我们来证 $W\subseteq B_k$,若 $d(b)\leqslant 2n+1$,则 $b\in W_{2n+1}\subseteq B_k$.设 $d(b)>2n+1$.并假定 W 中所有长度小于 $d(b)$ 的字都属

① 当正负号对我们的讨论没有什么影响的时候,为了不分散注意力,我们将用 ± 来代替 + 或 −.

于 B_k.

若在字 b 中只出现一因子是形如 x_i 者，假定这因子是 x_1，则只需讨论下面这种情形：字 b 是由顺序用形如 a_i 之元从左或从右去乘 x_1 的办法所得到的. 因为，不然的话，就会找到一个只含形如 a_i 因子的括号，那时 b 将是较 $d(b)$ 为短的带系数的 w 中之字的和，因而由假设知 b 是属于 B_k 中的了. 由于 $d(b)>2n+1$，故至少有一个形如 a_i 的，说是 a_1，在 b 中至少出现三次，利用变换(2)，(3)，可把 b 写成一些较短的字和一些具有形如

$$a_1(b_1a_1) \quad 或 \quad (a_1b_1)a_1,$$

其中 $d(b_1)\geqslant 2$，因子的字的和. 由(4) 知 $b=\backsim$. 即 $b\in B_k$.

若在字 b 中出现两个以上形如 x_i 之因子时，则和上面谈的一样，b 可看作是由顺序用形如 a_i 之元从左或从右去乘 b_1b_2 的办法得到的，由于

$$a(b_1b_2)=\pm(ab_1)b_2\pm(b_1a)b_2\pm b_1(ab_2),$$

$$(b_1b_2)a=\pm b_1(b_2a)\pm(b_1a)b_2\pm b_1(ab_2),$$

我们得 $b=\sum\limits_i \pm b_{i_1}b_{i_2}$，因为 $d(b_{ij})<d(b)$，故，由假设，所有的 $b_{ij}\in B_k$，即 $b\in B_k$.

推论 任意可解(因之，幂零) 交错代数 A 是局部有限的①.

证明：我们有

$$A=A^{(1)}\supset A^{(2)}\supset\cdots\supset A^{(i)}\supset\cdots\supset A^{(n)}=0.$$

由于每一 $A^{(i+1)}$ 都是 $A^{(i)}$ 之理想子代数，且商代数 $\dfrac{A^{(i)}}{A^{(i+1)}}$ 是零代数，因而是局部有限的. 故根据预理，立得这个推论.

预理 3 若基本域之特征数不为 2，则由局部有限代数借助于局部有限代数所得的扩张若当代数是局部有限的.

证明：我们在 A 中有

$$[(xy)w]z+[(xz)w]y+[(yz)w]x$$
$$=(xy)(wz)+(xz)(wy)+(yz)(wx).$$

① A 是任意代数(指非结合代数)，X，Y 是 A 之子代数，用 XY 表示所有形如 xy 之元素，其中，$x\in X$，$y\in Y$，在 A 中所生之子代数. 定义 $A^{(1)}=A$，$A^{(i+1)}=A^{(i)}A^{(i)}$；$A^1=A$，$A^{i+1}=A^iA$. 若存在自然数 n，使 $A^{(n)}=0$，则 A 被称为是可解的，若存在自然数 n，使 $A^n=0$，则 A 被称为是幂零的.

令 $x=b, y=a_i, w=a_j, z=a_k$. 得

$$[(b\,a_i)\,a_j]a_k = -[(a_i a_k)\,a_j]b - [(b\,a_k)\,a_j]a_i + (b\,a_i)(a_j a_k) + (a_i a_k)(a_j b) + (b\,a_k)(a_j a_i)$$
$$= -[(b\,a_k)\,a_j]a_i + \backsim. \tag{5}$$

若 $a_i = a_k$ 则 $[(b\,a_i)a_j]a_i = -[(b\,a_i)a_j]a_i + \backsim$, 因为基本域之特征数不等于 2, 故由是得

$$[(b\,a_i)a_j]a_i = \backsim. \tag{6}$$

设 $k=[2(n+1)]^2+2(n+1)$. 今对字 b 之长作归纳法来证 $W\subseteq B_k$. 若 $d(b)\leqslant k$, 则 $b\in W_k\subseteq B_k$. 设 $d(b)>k$.

若在字 b 中只有一个形如 x_i 之因子, 说是 x_1, 则由于若当代数是交换的, 得 $b=xa_{i_1}\cdots a_{i_t}$①, 因为不然的话 $b=\backsim$. 由于 $d(b)>k$, 故某一 a_i, 譬如说是 a_1, 必至少在 b 中出现三次. 利用交换(5) 我们得

$$b=\pm\, x\,a_i a_1\,a_j a_1\cdots+\backsim \quad 或 \quad b=\pm\, x\,a_1\,a_i a_1\cdots+\backsim,$$

其中 a_i, a_j 并不假定是和 a_1 不同的元素. 由(6) 知, $b=\backsim$, 即 $b\in B_k$.

现在来考察在字 b 中至少出现两次形如 x_i 之因子, 先提出两点注意事项.

若给出字 $b'=b_1b_2\,a_{i_1}\cdots a_{i_p}$, 其中 $p\geqslant 2$, 则由

$$b_1b_2\,a_ia_j = -(b_1\,a_ja_i)b_2 - (b_2\,a_ja_i)b_1 + (b_1\,b_2)(a_ia_j) + (b_1\,a_i)(b_2\,a_j) + (b_1\,a_j)(b_2\,a_i)$$
$$= \sum_{i=1}^{4} b_{i_1}b_{i_2} + \backsim.$$

得

$$b' = \sum_j b_{j_1}b_{j_2} + \backsim, \qquad 当\ p=2h,$$
$$b' = \sum_j b_{j_1}b_{j_2}\,a_{i_p} + \backsim, \qquad 当\ p=2n+1.$$

若字 b 有一形如 $b_1b_2b_3$ 之子字, 则由

$$(b_1b_2b_3)d = -[(b_1d)b_3]b_2 - (b_2d)b_3]b_1 + (b_1b_2)(b_3d) + (b_1b_3)(b_2d) + (b_2b_3)(b_1d)$$
$$= \sum_i b_{i_1}b_{i_2}b_{i_3},$$

① 其中 $x\,a_1\cdots a_{n-1}\,a_n = (x\,a_1\cdots a_{n-1})a_n$.

其中 $d = a_i$ 或 $d \in w$，得 $b = \sum_j b_{j_1} b_{j_2} b_{j_3}$，因之 $b \in B_k$.

若 $b = b_1 b_2$，则显然 $b \in B_k$. 因此根据上面提出的两点，剩下来的只需考察当字 b 有下面的形状：在 b 中有 S 个形为 x_i 之因子. 因此在 b 中可划出 S 个最大的括弧. 他们中的每一个只包含一个形为 x_i 之因子，而其他的因子是形为 a_i 的. 在这些括弧之外还剩下 t 个形为 a_i 之因子. 这些因子我们称之为自由的因子. 每一个自由因子 a_j 都只出现在形如 $b_1 b_2 a_j$ 之括弧中，而这个括弧不再被任何 a_k 去乘.

对 $d(b)$ 作归纳法，易证对这样的字. 有

$$\left[\frac{S}{2}\right] \leqslant t < S, \tag{7}$$

其中 $\left[\frac{S}{2}\right]$ 是 $\frac{S}{2}$ 的整数部分. 证明时只要注意到字 b 本身之形也是 $b = b_1 b_2 a_j$.

提醒一下

$$d(b) > k = [2(n+1)]^2 + 2(n+1). \tag{8}$$

现分下列两种情形来讨论：

a) 当 $S \leqslant 2(n+1)$ 时. 由 (7)，(8) 知字 b 中必有一子字 $b' = x_i a_{i_1} \cdot a_{i_2} \cdots a_{i_p}$，它之长度 $d(b') \geqslant 2n+2$. 因之，在这一子字中某一 a_i 必至少出现三次. 由刚才证过的知 $b = \backsim$ 即 $b \in B_k$.

b) 当 $S > 2(n+1)$ 时. 由(7)，(8) 知 $t \geqslant n+1$. 即在自由因子中至少有两个相同的.

$$\begin{aligned} b_1 b_2 a_i b_3 a_j &= -b_1 b_2 a_j b_3 a_i - (a_j a_i b_3)(b_1 b_2) + [(b_1 b_2) a_i](b_3 a_j) + \\ &\quad [(b_1 b_2) a_j](b_3 a_i) + (a_i a_j)(b_1 b_2 b_3) \\ &= -b_1 b_2 a_j b_3 a_i + b_1' b_2' + b_3' b_4' + \backsim, \end{aligned} \tag{9'}$$

$$(b_1 b_2 a_i b_3 a_j) b_4 a_k \cdots = -(b_1 b_2 a_j b_3 a_i) b_4 a_k \cdots + \sum_j b_{j_1} b_{j_2} b_{j_3} + \backsim, \tag{9''}$$

$$\begin{aligned} [(b_1 b_2) a_i][(b_3 b_4) a_j] &= -[(b_1 b_2) a_j][(b_3 b_4) a_i] - [(b_1 b_2)(b_3 b_4)] \cdot \\ &\quad (a_i a_j) + [(b_1 b_2) a_j](b_3 b_4) a_i + \\ &\quad [(b_1 b_2) a_i (b_3 b_4)] a_j + (a_i a_j)(b_3 b_4)(b_1 b_2) \\ &= -[(b_1 b_2) a_j][(b_3 b_4) a_i] + \sum_j b_{j_1} b_{j_2} b_{j_3} + \backsim. \end{aligned} \tag{9'''}$$

当 $a_i = a_j$，注意到基本域之特征数不是 2，得

$$I_1 = b_1 b_2 a_i b_3 a_i = \sum_j b_{j_1} b_{j_2} + \backsim, \tag{10'}$$

$$I_2 = (b_1 b_2 a_i b_3 a_i) b_4 a_k \cdots = \sum_j b_{j_1} b_{j_2} b_{j_3} + \backsim, \tag{10''}$$

$$J = [(b_1 b_2) a_i][(b_3 b_4) a_i] = \sum_j b_{j_1} b_{j_2} b_{j_3} + \backsim. \tag{10'''}$$

利用变换(9)，可将字 b 化为一些含形如I_1, I_2, J 子字之字的和. 因此利用 (10) 和上面谈到的一些注意事项，得 $b \in B_k$.

推理 若基本域之特征数不为2，则可解(因之，幂零) 若当代数是局部有限.

证明和证预理 2 之推理相仿.

至于谈到李代数，则我们给出一个例子来说明，这样的论断一般讲是不对的.

例 令 $e_i (i = 0, 1, 2, \cdots)$ 是代数 A 之基底，令

$$e_i^2 = 0, \quad i = 0, 1, 2, \cdots \quad e_i e_j = -e_j e_i, \quad i, j = 0, 1, 2, \cdots$$

$$e_i e_j = 0, \quad i, j = 1, 2, \cdots \quad e_0 e_i = e_{i+1}, \quad i = 1, 2, \cdots$$

易检验，A 是李代数，若 R 是以$e_i (i = 1, 2, \cdots)$ 为基底的向量空间，则 R 是 A 之局部有限理想子代数且 $\dfrac{A}{R}$ 是一维代数，但 A 不是局部有限的，因为 e_0, e_1 已生成 A.

由此例可见，可解李代数，一般言，也不见得是局部有限的. 并可看出预理 3 中对域之特征数之限制是必要的，因为上例中之代数当域之特征数等于 2 时也是若当代数.

我们在下面给出被讨论的扩张李代数是局部有限的充要条件，为了这个目的引入代数的李代数的概念，令 R 是李代数A 之理想子代数. 我们称 $a \in A$ 是 R 上的代数元素，若对任意 $x \in R$，存在这样一个正整数 $k = k(x, a)$，使下列元素：$x, xa, xa^2 = (xa)a, \cdots, xa^k = (xa^{k-1})a$ 是线性相关的. 称 A 在其理想子代数 R 上是代数的，若其每一元素是 R 上之代数元素. 称 A 是代数的李代数，若 A 在自己上(在 A 上) 是代数的.

预理 4 由局部有限代数 R 借助于局部有限代数所得之扩张李代数 A 是局部有限的充要条件是 A 在其理想子代数 R 上是代数的.

证明：若 A 是局部有限的，则 A 之任二元素在其中生成有限子代数.

因之 A 在 R 上是代数的.

若 A 在 R 上是代数的,则对每个 $a_i(i=1,2,\cdots,n)$ 存在一自然数 n_i,有

$$xa_i^{n_i}=\backsim, \tag{11}$$

其中 x 是 R 之子代数X 之任意元素.令 $k=\sum_{i=1}^{n} n_i+1$.用对字之长作归纳法来证 $\omega\subseteq B_k$.

在 A 中我们有:$(ab)c+(bc)a+(ca)b=0$.令 $a=a_i,c=a_j$.得

$$(ba_j)a_i=\pm(a_ia_j)b\pm(ba_i)a_j=\pm(ba_i)a_j+\backsim. \tag{12}$$

若 $d(b)\leqslant k$,则 $b\in\omega_k\subseteq B_k$.设 $d(b)>k$.

若在字 b 中只有一个形为x_i 之因子,说是 x,由于 $cd=-dc$,同在前面所讨论的一样(见预理 3 之证明),可设

$$b=xa_{i_1}\cdots a_{i_p}.$$

因为 $d(b)>k=\sum_{i=1}^{n} n_i+1$,故某一 a_i,说是 a_1,在 b 中至少出现n_1 次.利用变换(12) 得

$$b=\pm xa_1^{n_1}a_{i_1}\cdots a_{i_t}+\backsim.$$

由之,据(11),得 $b=\backsim$,即 $b\in B_k$.

假若在 b 中出现两个以上的形如x_i 之因子,则 $b=b_1b_2a_{i_1}\cdots a_{i_p}$.由等式:$(b_1b_2)a=\pm(b_1a)b_2\pm(b_2a)b_1$,立得 $b=\sum\pm b_ib_{i_2}\in B_k$.

推理 1 代数的可解李代数 A 是局部有限的.

这是我们附带得到的与李代数中 Engel 问题相关的结果(参看[13]).这个推论的证明和上面的推理的证明相同.

推理 2 幂零李代数是局部有限的.

这是因为幂零李代数是代数的可解李代数,我们把这可以直接的、很易证明的结果作为一个推理放在这里是因以后要用到.

§3.一些定义和符号

设在代数 A 中被选出理想子代数串$\{R_i\}$,有性质

$$R=R_1\supset R_2\supset\cdots\supset R_n\supset\cdots,\quad \bigcap_{n=1}^{\infty}R_n=0.$$

若把R_i，$i = 1,2,\cdots$当作是零的邻域，则A将成为拓扑代数. 用$\{R_i\}$所定义的拓扑结构叫做R-拓扑结构.

我们称域Φ上的拓扑向量空间M是在拓扑结合(交错，李，若当)代数A(在域Φ上的)上的拓扑结合(交错，李，若当)模，若被定义了一个M之元素与A之元素间的乘法，满足下面条件：

1. $MA \subseteq M$，$AM \subseteq M$.

2. 若$\alpha \in \Phi$，$m, m_1, m_2 \in M$，$a, a_1, a_2 \in A$，则

$$\alpha(ma) = (\alpha m)a = m(\alpha a);\quad \alpha(am) = (\alpha a)m = a(\alpha m);$$

$$(a_1 + a_2)(m_1 + m_2) = a_1m_1 + a_1m_2 + a_2m_1 + a_2m_2;$$

$$(m_1 + m_2)(a_1 + a_2) = m_1a_1 + m_1a_2 + m_2a_1 + m_2a_2.$$

3. 在结合模的情形，有恒等式$(xy)z = x(yz)$，其中x, y, z三元素中的一个属于M，其余的属于A.

在交错模的情形，表示式$[x, y, z] = (xy)z - x(yz)$经其中任二元素之对换后变号，其中$x, y, z$三元素中的一个属于$M$，其余的属于$A$.

在李模的情形，$-am = ma$，其中$a \in A$，$m \in M$，还有$(xy)z + (yz)x + (zx)y = 0$，其中$x, y, z$三元素中的一个属于$M$，其余的属于$A$.

在若当模的情形，$am = ma$，其中$a \in A$，$m \in M$，$[(xy)w]z + [(xz)w]y + [(yz)w]x = (xy)(wz) + (xz)(wy) + (yz)(wx)$，其中$x, y, z, w$四元素之一属于$M$，其余的属于$A$.

4. M中之元素和A中元素间的乘法是连续的. 即若给定$am \in M$($ma \in M$)，其中$a \in A$，$m \in M$之任一邻域N，则可找出元素a之邻域A_1和元素m之邻域M_1，使有$A_1M_1 \subseteq N$($M_1A_1 \subseteq N$).

为简便计，我们将用A-模来代替."在代数A上的模"，称M中之自成为A-模的封闭子向量空间为A-模M的A-子模. 若$m \in M$，M_1是M的A-子模，则称集合$m + M_1$为线性集合. 一个A-模M被称为是线性紧致的，若具有下性质(将称之为有限交)的由线性集合组成之集合的所有元素之交不为零. 这个集合的每有限元素(线性集合)之交不为零.

为简便计，把符号的用法肯定下来，用A表下列代数：在任意域上的结合代数，在特征数为零的域上的交错代数，或若当代数，或代数的李代数. 我们将不重复对域的特征数的限制，只是说"在特征数为被允许的域

上的代数 A". 字母 N,C,P,Q,R 之用法请看[1] 中 §4,"A 是可分解的", R - 内自同构, R - 共轭, R - 弱共轭等之意义亦请看[1] 中 §4.

对结合代数,交错代数,李代数. 令

$$R^{[1]} = R, \quad R^{[k+1]} = \{RR^{[k]}, R^{[k]}R\}$$

$$R_1 = R, \quad R_{k+1} = R_kR.$$

显然,对结合代数和李代数言 $R^{[k]} = R_k$. 对若当代数令

$$R_1 = R, \quad R_k = \{R_{k-1}R^2, (R_{k-1}R)R\}$$

$$R^{[k]} = R_k,$$

因为 R 是 A 之理想子代数,故 $R_k(R^{[k]})$ 也是 A 之理想子代数. 事实上,当讨论结合,交错,李代数时,欲证这一事实,只需利用 Jacobi 恒等式,或关于结合子(指$[a,b,c]$)之恒等式(参看预理 2 证明的开始部分)很易证得,至于谈到若当代数,设 B_1,B_2,B_3 是若当代数 A 之三个理想子代数,令

$$B = \{B_1(B_2B_3), B_2(B_1B_3), B_3(B_1B_2)\},$$

则 B 将是理想子代数,这是因为

$$[(B_1B_2)B_3]A = \pm[(B_1A)B_3]B_2 \pm [(B_2A)B_3]B_1 \pm (B_1B_2)(B_3A) \pm (B_1A)(B_3B_2) \pm (B_2A)(B_3B_1).$$

§4. 局部 N - 根

关于局部 N - 代数,局部 N - 理想子代数和局部 N - 根之定义可参看[1].

预理 5　若 B 是结合代数,或交错代数,或代数的李代数或若当代数 A(其域之特征数是被允许的) 之局部 N - 理想子代数. 若商代数$\frac{A}{B}$ 是局部N - 代数,则 A 也是局部N - 代数.

证明: 由预理 1 ~ 4 和由他们得到的推论,知代数 A 是局部有限的. 若 D 是A 的有限子代数,则 $B \cap D$ 和$\frac{D}{B \cap D}$ 也是有限N - 代数. 因之, D 本身也是 N - 代数.

推论　若 R 是代数A 中所有的局部N - 理想子代数之和,则 R 是局部 N - 理想子代数而$\bar{A} = \frac{A}{R}$ 将不再有非零的局部N - 理想子代数.

这一推论说明：在我们的四种代数中局部 N - 根是存在的.

根据 §2 的结果，由[1] 中之定理 1 可导出下面定理. 在定理的叙述中对结合，交错，若当代数的情形略去了 A 是局部有限的假设，而对李代数的情形，则用"A 是代数的" 的假设去代替条件"A 是局部有限的".

定理 1 令 R 是结合代数，或交错代数，或若当代数或代数的李代数 A（其域之特征数是被允许的）之局部 N - 根，若$\frac{A}{R}$是局部C - 代数且其维数不多于可数的话，则

(i) A 是可分解的，即 $A = P + R$.

(ii) Q 在A 中与P 之某一子代数弱R - 共轭.

这里我们要特别提出下面这一点来. 就是在以后我们将用到 R - 内自同构的具体表示式$1 + f$，在研究[6]，[7]，[8]，[10] 中关于这四种情形有限代数的分解的唯一性（对 R - 内自同构言）的定理以后，并且注意到，事实上在[1] 中之预理2 和定理1 中所出现的R - 内自同构是代数A 中之某一有限子代数中之可扩充到 A 上的N - 内自同构，我们能断定它们的表示式是有下面形状的.

a) 对结合代数言，$\sigma = 1 - R_x - L_{x'} + L_{x'}R_x$，　其中 $xx' = x'x = x + x'$，$x, x' \in R$；

b) 对交错代数言，$\sigma = \prod_{j=1}^{n} \exp D_j$①，其中 D_j 之形如D.

$$D = \sum_{i=1}^{m} D_{y_i x_i}$$

$$= \sum_{i=1}^{m} ([R_{y_i}, R_{x_i}] + [L_{y_i}, R_{x_i}] + [L_{y_i}, L_{x_i}]), \quad y_i \in A, x_i \in R.$$

c) 对若当代数言，$\sigma = \prod_{j=1}^{n} \exp D_j$，其中 D_j 之形如D.

$$D = \sum_{i=1}^{m} D_{y_i x_i} = \sum_{i=1}^{m} [R_{y_i}, R_{x_i}], \quad y_i \in A, x_i \in R.$$

d) 对李代数言，$\sigma = \prod_{j=1}^{n} \exp D_j$，其中 $D_j = R_{x_j}$，$x_j \in R$.

其中$[a, b] = ab - ba$.

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① 其中 $\exp D = 1 + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots$

§5. 预　　理

预理6　令 R 是结合代数，或交错代数，或若当代数或代数的李代数 A（其域的特征数是被允许的）之局部 N-根. 若 $R^2=0$ 和 R 是线性紧致（对离散拓扑结构言）的 A-模，若 $\bar{A}=\dfrac{A}{R}$ 是任意多个有限 C-代数的直和，则 A 是可分解的.

证明：根据预理 1 ~ 4 及其推论，由本预理之假设立即得 A 是局部有限的.

令 $\bar{A}=\sum\limits_{\alpha}\bar{A}_\alpha$，其中 α 走遍某一脚码的集合 w，令有限 C-代数 $\bar{A}_\alpha$ 之基底是 $\bar{b}_{\alpha_1},\cdots,\bar{b}_{\alpha_{n_\alpha}}$. 若 u 是 w 之一子集合，则用 $\bar{B}_u$ 表 $\sum\limits_{\alpha\in u}\bar{A}_\alpha$，用 B_u 表子代数 $\bar{B}_u$ 在 A 中的完全原像. 对所有可分解的子代数 B_u 之集合应用 Zorn 的选择公理，我们可得到子代数 B_v，其分解式

$$B_v=P_v+R \tag{13}$$

是不可以继续的，这是说，不存在 w 之子集合 $u\supset v$，其所相应的 B_u 之分解式 $B_u=P_u+R$ 有性质 $P_u\supset P_v$. 现在剩下来要证明的是 $v=w$，即要证 $B_v=A$.

假设存在一脚码 $\beta\notin v$. 令 $P_v=\sum\limits_{\alpha\in v}P_\alpha$，其中 P_α 之基底设为 a_{α_i}，$i=1,2,\cdots,n_\alpha$，并且 $\bar{b}_{\alpha_i}=\bar{a}_{\alpha_i}$①.

我们先证，在每一 R 之陪集 $b_{\beta_i}\in\bar{A}_\beta$ 中存在一零化 P_v 之代表. 令 a 是陪集 $\bar{a}\in\bar{A}_\beta$ 某一固定的代表. 我们作下面对于未知元素 x（在 R 中）之方程组 (G_α)，$\alpha\in v$：

$$(a-x)a_{\alpha_i}=0 \quad 即 \quad xa_{\alpha_i}=aa_{\alpha_i},$$

$$a_{\alpha_i}(a-x)=0 \quad 即 \quad a_{\alpha_i}x=a_{\alpha_i}a,$$

其中 $i=1,2,\cdots,n_\alpha$. 易见，若此方程组之解 x 存在，则在陪集 $\bar{a}$ 中可找到零化 P_α 之代表，反之亦然.

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① 若 B 是代数 A 的子集合，则 $\bar{B}$ 表示在同态映像 $A\to\bar{A}=\dfrac{A}{R}$ 下 B 在 $\bar{A}$ 中之像.

若 x_1, x_2 是方程 $xa_{\alpha_i} = aa_{\alpha_i}(a_{\alpha_i}x = a_{\alpha_i}a)$ 之解，则 $x_1 - x_2$ 是方程 $xa_{\alpha_i} = 0(a_{\alpha_i}x = 0)$ 之解.

今证，方程组(G_α')：

$$xa_{\alpha_i} = 0 \quad 和 \quad a_{\alpha_i}x = 0, i = 1,2,\cdots,n_\alpha,$$

之解组成代数 A 的理想子代数，为此只需证

$$a_{\alpha_i}(bx) = (bx)a_{\alpha_i} = a_{\alpha_i}(xb) = (xb)a_{\alpha_i} = 0, \tag{14}$$

其中 $b \in A$，而 x 是方程组(G_α') 之解. 令 a_α 表a_{α_i} 中任意元素，易见

$$ba_\alpha = \sum f_i a_{\alpha_i} + x_\alpha' \quad a_\alpha b = \sum f_i' a_{\alpha_i} + x_\alpha'',$$

其中 $x_\alpha', x_\alpha'' \in R$，而 f_i 和f_i' 属于基本域Φ 中. 因为 $R^2 = 0$，故有

$$x(ba_\alpha) = x(a_\alpha b) = (ba_\alpha)x = (a_\alpha b)x = 0. \tag{15}$$

我们现在对每一种情形分别来证等式(14).

a) 对结合代数言，由 (15) 和 x 之选择知 (14) 是显然成立的.

b) 对交错代数言，有

$$a_\alpha(bx) = \pm(a_\alpha b)x \pm (ba_\alpha)x \pm b(a_\alpha x) = 0,$$

$$同样\ a_\alpha(xb) = (bx)a_\alpha = (xb)a_\alpha = 0.$$

c) 对李代数言，有

$$a_\alpha(bx) = \pm(a_\alpha b)x \pm (a_\alpha x)b = 0,$$

$$因之\ a_\alpha(xb) = (bx)a_\alpha = (xb)a_\alpha = 0.$$

d) 对若当代数言. 因为 P_α 是半单纯的，易见 $a_\alpha = \sum f_{ij}a_{\alpha_i}a_{\alpha_j}$. 因之为了证 (14)，只需证 $(a_{\alpha_i}a_{\alpha_j})(bx) = 0$ 就够了，我们来做这一点.

$$\begin{aligned}(a_{\alpha_i}a_{\alpha_j})(bx) &= \pm(a_{\alpha_i}x)(a_{\alpha_j}b) \pm (a_{\alpha_i}b)(a_{\alpha_j}x) \pm [(a_{\alpha_i}a_{\alpha_j})b]x \pm \\ &\quad [(a_{\alpha_i}x)b]a_{\alpha_j} \pm [(a_{\alpha_j}x)b]a_{\alpha_i} \\ &= 0.\end{aligned}$$

这样我们证明了，方程组(G_α') 之解的全体组成 A 之理想子代数E_α. 因之，方程组(G_α) 之解的全体组成线性集合 $x_\alpha + E_\alpha$，其中 x_α 是方程组 (G_α) 的任一解.

可断言，所有线性集合 $x_\alpha + E_\alpha, \alpha \in v$，所成之集合 E 必有有限交的性质，即 E 之任意有限元素(线性集合) 之交不空，事实上，由定理 1 知对任意有限个脚码 $a_i \in v, i = 1,2,\cdots,n$. 子代数 $\bar{H} = \bar{A}_\beta + \sum_{i=1}^{n}\bar{A}_{\alpha_i}$ 在A 中

之完全原像 H 是可分解的，即 $H = P_{\beta\alpha_i} + R$，并且可以适当的选择 $P_{\beta\alpha_i}$，使有 $\sum_{i=1}^{n} P_{\alpha_i} \subset P_{\beta\alpha_i}$. 因之，在每一陪集 $\bar{a} \in \bar{A}_\beta$ 中存在一零化 $\sum_{i=1}^{n} P_{\alpha_i}$ 的代表. 即已证，集合 E 是有有限交的性质的.

由于根 R 是线性紧致的，故集合 E 中之所有元素有不空之交，即对任意陪集 $\bar{a} \in \bar{A}_\beta$，存在有零化 P_v 之代表 a，即 $aP_\alpha = P_\alpha a = 0$，其中 α 是属于 v 中的任意脚码. 若在每一陪集 $\bar{b}_{\beta_i} \in A_\beta$ 中取一零化 P_v 之代表，则这些代表组成一有限集合. 因 A 是局部有限的，这集合生成一有限子代数 $A_\beta \subseteq A$. 今证

$$A_\beta P_v = P_v A_\beta = 0.$$

为此，只需证：若 $s \in P_\alpha$，其中 $\alpha \in v$，且知 $P_\alpha a = aP_\alpha = P_\alpha b = bP_\alpha = 0$，则 $s(ab) = (ab)s = 0$. 重复上面所引入之讨论 a) ~ d)，可证此论断.

由于 $\frac{A_\beta}{A_\beta \cap R} \simeq \bar{A}_\beta$，据定理 1 得

$$A_\beta = P_\beta + A_\beta \cap R,$$

其中 $\bar{P}_\beta = \bar{A}_\beta$ 且知 $P_\beta P_v = P_v P_\beta$，若 $B_{v\cup\beta}$ 是子代数 $B_v + \bar{A}_\beta$ 在 A 中之完全原像，则

$$B_{v\cup\beta} = (P_v + P_\beta) + R.$$

这和分解式(13) 之选择是矛盾的.

预理 7　令 R 是结合代数，或交错代数，或代数的李代数 A（其域的特征数是被允许的）的局部 N - 根. 若 R 是线性紧致（在离散拓扑结构中）A - 模，若 $\bar{A} = \frac{A}{R}$ 是任意多有限 C - 代数的直和，则对每一自然数 k 有下面等式：$A = S_k + R$，其中 S_k 是子代数，$S_k \cap R = R^{[k]}$ 而 $S_k \geqslant S_{k+1}$. 于是得若 $R^{[n]} = 0$，则 A 是可分解的.

证明： 设对某一自然数 k 有 $A = S_k + R$，其中 S_k 是子代数而 $S_k \cap R = R^{[k]}$（当 $k = 1$ 取 A 作 S_1，我们可得这样的等式）. 现来研究代数 $\frac{S_k}{R^{[k+1]}}$.

$$\frac{\frac{S_k}{R^{[k+1]}}}{\frac{R^{[k]}}{R^{[k+1]}}} \simeq \frac{S_k}{R^{[k]}} = \frac{S_k}{S_k \cap R} \simeq \frac{S_k + R}{R} = \frac{A}{R} = \bar{A}.$$

现来证:$\frac{R^{[k]}}{R^{[k+1]}}$ 是线性紧致$\left(\frac{S_k}{R^{[k+1]}}\right)$-模.

事实上,若令 $\frac{E}{R^{[k+1]}}$ 是代数 $\frac{S_k}{R^{[k+1]}}$ 的任意包含在$\frac{R^{[k]}}{R^{[k+1]}}$ 内的理想子代数$\left(\text{也是 }\frac{S_k}{R^{[k+1]}}\text{-模}\right)$,令 E 是 $\frac{E}{R^{[k+1]}}$ 在 A 中之完全原像,因 $R^{[k]} \supseteq E \supseteq R^{[k+1]}$,故有 $E \supseteq RR^{[k]} \supseteq RE, E \supseteq R^{[k]}R \supseteq ER, S_kE \subseteq E$ 和 $ES_k \subseteq E$,由之可得 E 是A 之理想子代数,因此 $\frac{S_k}{R^{[k+1]}}$-模 $\frac{S^{[k]}}{R^{[k+1]}}$ 中之线性集合 $x+\frac{E}{R^{[k+1]}}$ 在 A 之完全原像也是A-模R 之线性集合,故最后得 $\frac{R^{[k]}}{R^{[k+1]}}$ 是线性紧致 $\frac{S_k}{R^{[k+1]}}$-模.

但知

$$\left(\frac{R^{[k]}}{R^{[k+1]}}\right)^2 = 0,$$

故,据预理 6,有

$$\frac{S_k}{R^{[k+1]}} = \frac{S_{k+1}}{R^{[k+1]}} + \frac{S^{[k]}}{R^{[k+1]}},$$

$$\text{即} \quad S_k = S_{k+1} + R^{[k]},$$

其中 S_{k+1} 是子代数$\frac{S_{k+1}}{R^{[k+1]}}$ 在 A 中之完全原像,且知 $S_{k+1} \cap R^{[k]} = R^{[k+1]}$,由是得

$$A = S_k + R = S_{k+1} + R^{[k]} + R = S_{k+1} + R.$$

其中 $S_{k+1} \cap R = S_{k+1} \cap S_k \cap R = R^{[k]} \cap S_{k+1} = R^{[k+1]}$.至此我们证明了对每一自然数 k 都有等式$A = S_k + R$,其中 S_k 是子代数而 $S_k \cap R = R^{[k+1]}$.

若当 $R^{[n]} = 0$,则得 $A = S_n + R$,其中 $S_n \cap R = R^{[n]} = 0$.

§6.预　　理

预理8　设 A 是结合代数,或交错代数,或若当代数,或李代数,B 是离散拓扑结构下的 A-模,则对 B 之A-子模有最小条件与A-模B 是线性紧致是等价的.

证明:设 E 是具有有限交性质的,由线性集合 $x_\alpha + E_\alpha$,其中 $\alpha \in W$,

而 W 是脚码的某一集合组成的集合，E 中任意有限个元素之交仍是线性集合，显然是不空的集合. 所有可能这样得来的线性集合用 $x_\beta' + E_\beta'$ 来表示，其中 $\beta \in W'$，而 W' 是脚码的某一集合. 若对 B 之 A - 子模有最小条件，则在 A - 模 E_β'，$\beta \in W'$ 中至少存在一个最小的，说是 E_1'. 令 $x_\alpha'' + E_\alpha'' = (x_1' + E_1') \cap (x_\alpha + E_\alpha)$. 对任意 $\alpha \in W$ 言 $X_\alpha'' + E_\alpha''$ 是不空的，因之 $E_\alpha'' = E_1' \cap E_\alpha = E_1'$. 因此 $x_\alpha + E_\alpha \supseteq x_1' + E_1'$. 由是可见 E 有不空之交，即由最小条件得出来线性紧致性.

现在来证反方向的论断. 设在 B 中对 A - 子模不满足最小条件. 那就必存在由 A - 子模组成的无限叙列 $\{M_n\}$，有性质

$$M_1 \supset M_2 \supset \cdots \supset M_n \supset \cdots$$

设 x_i 是某一属于 M_i 而不属于 M_{i+1} 之元素，显然元素 $x_i (i = 1, 2, \cdots)$ 是线性无关的. 因之，可以把它们看作是 M_1 的底元. 令由线性集合 $\sum_{i=1}^{n} x_i + M_{n+1}$，$n = 1, 2, \cdots$ 所组成的集合数 E. 易见 E 是具有有限交性质的. 若 B 是线性紧致的 A - 模，则 E 有不空之交. 即至少存在一元素 b 属于所有的线性集合 $\sum_{i=1}^{n} x_i + M_{n+1}$，$n = 1, 2, \cdots$ 元素 b 应该可表示为 M_1 的有限个基元的线性组合，但这却和元素 b 之定义不合.

预理 9 令 R 是结合代数或交错代数 A（在被允许的特征数的域上）的局部 N - 根. 若 $R^2 = 0$，及 R 是在离散拓扑结构中线性紧致的 A - 模. 且若 $\bar{A} = \frac{A}{R}$ 是任意多有限 C - 代数的直和，则 P' 在 A 中与 P 共轭. 说得更清楚一些，$P\sigma = P'$，而 σ 之表示式在 §4 中曾给出了.

证明：设 $P = \sum_\alpha S_\alpha$，$P' = \sum_\alpha S_\alpha'$，$\bar{S}_\alpha = \bar{S}_\alpha'$，$\alpha \in W$，而 W 是脚码的某一集合，令 e_α 是半单纯代数 S_α 之单位元.

先证：$S_\alpha R$ 是 A 之理想子代数，因 $R^2 = 0$，故为此只需证 $(S_\alpha R) S_\beta$ 及 $S_\beta(S_\alpha R)$，$\beta \in W$，包含在 $S_\alpha R$ 中.

a) 对结合代数言这是显然成立的.

b) 对交错代数言，由于当 $\alpha \neq \beta$ 时，$S_\alpha S_\beta = S_\beta S_\alpha = 0$. 故对任一 $\beta \in W$ 我们有

$$S_\beta(S_\alpha R) = \pm (S_\beta S_\alpha) R \pm (S_\alpha S_\beta) R \pm S_\alpha(S_\beta R) \subseteq S_\alpha R,$$

$$(S_\alpha R) S_\beta = \pm S_\alpha (R S_\beta) \pm (S_\alpha S_\beta) R \pm S_\alpha (S_\beta R) \subseteq S_\alpha R.$$

故证得 $S_\alpha R$ 是 A 之理想代数，下面我们来证，当 $\alpha \neq \beta$ 时 $S_\alpha R \cap S_\beta R = 0$. 为此只需证 $e_\alpha(s_\alpha x) = S_\alpha x, e_\alpha(s_\beta x) = 0$，其中 $s_\alpha \in S_\alpha, s_\beta \in S_\beta$，$\alpha \neq \beta$ 而 $x \in R$. 这是因为由于 $R^2 = 0$，$S_\alpha R$ 中之元素是形如 $S_\alpha x$ 之和的缘故.

a) 对结合代数言这是显然成立的.

b) 对交错代数言，有

$$e_\alpha(s_\gamma x) = (e_\alpha s_\gamma)x - [e_\alpha, s_\gamma, x] = (e_\alpha s_\gamma)x + [s_\gamma, e_\alpha, x]$$
$$= (e_\alpha s_\gamma)x + (s_\gamma e_\alpha)x - s_\gamma(e_\alpha x),$$

若 $\gamma = \alpha$，则 $\qquad e_\alpha(s_\alpha x) = 2s_\alpha x - s_\alpha(e_\alpha x), \qquad (16)$

若 $\gamma = \beta$，则 $\qquad e_\alpha(s_\beta x) = - s_\beta(e_\alpha x). \qquad (17)$

利用(16) 可得

$$e_\alpha(s_\chi x) = e_\alpha[e_\alpha(s_\alpha x)] = e_\alpha[2s_\alpha x - s_\alpha(e_\alpha x)]$$
$$= 2e_\alpha(s_\alpha x) - e_\alpha[s_\alpha(e_\alpha x)]$$
$$= 2e_\alpha(s_\alpha x) - \{(e_\alpha s_\alpha)(e_\alpha x) + (s_\alpha e_\alpha)(e_\alpha x) - s_\alpha[e_\alpha(e_\alpha x)]\}$$
$$= 2e_\alpha(s_\alpha x) - s_\alpha(e_\alpha x),$$

即 $e_\alpha(s_\alpha x) = s_\alpha(e_\alpha x)$. 因此，再利用(16)，得

$$e_\alpha(s_\alpha x) = s_\alpha x,$$

利用(17) 可得

$$e_\alpha(s_\beta x) = e_\alpha[e_\alpha(s_\beta x)] = - e_\alpha[s_\beta(e_\alpha x)]$$
$$= s_\beta[e_\alpha(e_\alpha x)] = s_\beta(e_\alpha x) = - e_\alpha(s_\beta x).$$

即 $e_\alpha(S_\beta x) = 0$. 这样证明了当 $\alpha \neq \beta$ 时，$S_\alpha R \cap S_\beta R = 0$.

同样 $\qquad \left(\sum_{i=1}^{m} S_{\alpha_i} R\right) \cap S_\beta R = 0, \alpha_i \neq \beta.$

相仿地可证：RS_α 是 A 的理想子代数，

且有 $\qquad \left(\sum_{i=1}^{m} RS_{\alpha_i}\right) \cap RS_\beta = 0, \alpha_i \neq \beta.$

假设存在无限多个非零的理想子代数 $S_\alpha R$，说是 $S_{\alpha_i} R, i = 1, 2, \cdots$ 若令 $R(n) = \sum_{i=n}^{\infty} S_{\alpha_i} R$，则得无限的真正下降的由理想子代数 $R(n) \subseteq R$ 作成之叙列，这是不可能的，因为据预理 8，对包含在 R 中之 A 的理想子代数言最小条件是成立的. 因此在 $S_\alpha R, \alpha \in W$ 中非零的只可能有有限个. 这说明除有限个脚码外，对 W 中之脚码 α 我们有 $S_\alpha R = RS_\alpha = 0$. 这样可

以把 P 和 P' 写成下面的形状,

$$P = P_1 + P_2, P' = P_1' + P_2',$$

其中 $\bar{P}_i = \bar{P}_i', i = 1,2. P_1$ 是有限 C - 代数而 $P_2R = RP_2 = 0$.

据定理 1 知,在代数 $A_1 = P_1 + R = P_1' + R$ 中存在 R - 内自同构 $\sigma = 1 + f(R_{x_i}; L_{y_j})$, 其中 $x_i, y_j \in A_1$, 使 $P_1\sigma = P_1'$. 就像在[1] 中那样 (参看[1] 中之预理 1 和定理 1 的证明), σ 也是 A 中的 R - 内自同构. 注意到 $P_2R = RP_2 = 0$ 及 $P_2P_1 = P_1P_2 = 0$, 得

$$P\sigma = (P_1 + P_2)\sigma = P_1' + P_2.$$

另一方面,若 $A_\alpha = S_\alpha + R = S_\alpha' + R$, 其中 $S_\alpha \subseteq P_2$. 则据定理 1 在 A_α 中存在一个 R - 内自同构 $\sigma_\alpha = 1 + f(R_{x_{\alpha_i}} L_{y_{\alpha_i}})$ 使 $S_\alpha\sigma_\alpha = S_\alpha'$. 因在 f_α 之每一项中必有一 x_{α_i} 或 y_{α_i} 属于 R, 而 $P_2R = RP_2 = 0$, 故得 $S_\alpha = S_\alpha'$, 即 $P_2 = P_2'$. 因此

$$P\sigma = P_1' + P_2 = P_1' + P_2' = P'.$$

预理 10 令 R 是代数的李代数 A (其域之特征数为 0) 中的局部 N - 根. 若 $R^2 = 0$, 且 $A = P + R = P' + R$, 其中 P, P' 是 A 中之有限 C - 代数, 则存在一 R - 内自同构 $\sigma = 1 + R_x$. 其中 $x \in R$, 使 $P\sigma = P'$.

这由定理 1 后面一段话立即推出.

预理 11 令 R 是代数的李代数 A (其域之特征数为 0) 中的局部 N - 根. 若 $R^2 = 0$, R 是在离散拓扑结构中线性紧致的 A - 模, 若 $\bar{A} = \dfrac{A}{R}$ 是任意多有限 C - 代数的直和, 则 P 在 A 中与 P' 是 R - 共轭, 即 $P\sigma = P'$, 其中 $\sigma = 1 + R_x, x \in R$.

证明: 令 $P = \sum_\alpha S_\alpha, P' = \sum_\alpha S_\alpha'$, 其中 $\bar{S}_\alpha = \bar{S}_\alpha', \alpha \in W$. 而 W 是脚码的某一集合. 令 S_α 及 S_α' 之底元顺序为 $s_{\alpha_i}; s_{\alpha_i}', i = 1,2,\cdots, n_\alpha. \bar{S}_{\alpha_i} = \bar{S}_{\alpha_i}'$. 今作对未知元素 x 之方程组 (G_α)

$$s_{\alpha_i} + s_{\alpha_i}x = s_{\alpha_i}', i = 1,2,\cdots, n_\alpha, \alpha \in W.$$

若 x_1, x_2 是 (G_α) 之解, 则 $x_1 - x_2$ 是下面方程组 (G_α') 之解:

$$s_{\alpha_i}x = 0, i = 1,2,\cdots, n_\alpha, \alpha \in W.$$

令 E_α 是方程组 (G_α') 在 R 中所有的解的集合. 由

$$(AE_\alpha)S_\alpha = \pm (E_\alpha S_\alpha)A \pm (S_\alpha A)E_\alpha \subseteq (S_\alpha + R)E_\alpha = 0.$$

知 $AE_\alpha \subseteq E_\alpha$，即 E_α 是 A 的理想子代数，显然，若 X_α 是方程组 (G_α) 的任一解，则 $X_\alpha + E_\alpha$ 是方程组(G_α) 在 R 中所有解的集合.

据预理 10 知，所有的线性集合 $X_\alpha + E_\alpha, \alpha \in W$，所组成的集合是具有有限交的性质的. 故由 A - 模 R 是线性紧致的立得，E 有不空之交，即存在元素 $X \in R$，对 W 中之任意 α 言，X 满足方程组 (G_α)，这说明

$$\left(\sum_{\alpha \in W} S_\alpha\right)(1 + x) = \sum_{\alpha \in W} S'_\alpha.$$

易见，$\sigma = 1 + R_x$ 是A 之R - 内自同构.

§7. 若当代数的分解

定理2 令 R 表若当代数A（在特征数为 0 的域上）的局部 N - 根，若 R 是有限的，而 $\bar{A} = \dfrac{A}{R}$ 是任意多有限 C - 代数的直和，则 A 是可分解的.

证明：先证：存在 A 之理想子代数R'，有性质 $R \supset R' \supseteq R^2$. 今取任意包含 R 的A 之有限子代数B. 由 [9] 中之预理知，有一由代数 B 之理想子代数$(R_k)_B$，$k = 1, 2, \cdots, n = n(B)$，组成的降叙列，其中

$$(R_0)_B = R, (R_{k+1})_B = B[(R_k)_B]^2 + [(R_k)_B]^2$$

且知

$$(R_n)_B \subseteq R^2.$$

若有 $R = (R_1)_B$，则 $R = (R_1)_B = \cdots = (R_n)_B \subseteq R^2$. 但 R 是幂零的，必 $R \supset R^2$. 由此得 $R \supset (R_1)_B$.

设 $R' = AR^2 + R^2$，则 R' 是A 之理想子代数（见 §3）且必有 $R \supset R' = AR^2 + R^2 \supseteq R^2$. 因为不然的话，由于 R 是有限的，必存在有限子代数 B，使得 $R = (R_1)_B$.

现对 R 之维数作归纳法来证明此定理. 由于 R 是有限的，据预理 8 知 R 在离散拓扑结构中是线性紧致的A - 模.

若 $R^2 = 0$（当 $n = 1$ 时，$R^2 = 0$），由预理 6 知定理是成立的.

设 $R^2 \neq 0$，我们来研究代数 $\dfrac{A}{R'}$. 显然 $\dfrac{R}{R'}$ 之维数小于R 之维数，且

$$\frac{\dfrac{A}{R'}}{\dfrac{R}{R'}} \simeq \frac{A}{R}.$$

由归纳法假设,有

$$\frac{A}{R'}=\frac{S'}{R'}+\frac{R}{R'},$$

即
$$A=S'+R,$$

其中 S' 是 $\frac{S'}{R'}$ 在 A 中的完全原像,易知 $S'\cap R=R'$. 因为 R' 之维数小于 R 之维数,且

$$\frac{S'}{R'}=\frac{S'}{S'\cap R}\simeq\frac{A}{R}.$$

故据同样的理由,我们有

$$S'=S+R',$$

其中 S 是子代数而 $S\cap R'=0$.

最后我们得 $A=S'+R=S+R$, 其中 $S\cap R=S\cap S'\cap R=S\cap R'=0$.

定理 3 令 R 表若当代数 A(在特征数为 0 的域上)的局部 N-根. 若 R 是有限的,且 $\bar{A}=\frac{A}{R}$ 是任意多有限 C-代数的直和,则 P 与 P' 在 A 中 R-共轭.

证明:令 $P=\sum_\alpha S_\alpha$, $P'=\sum_\alpha S'_\alpha$, $\bar{S}_\alpha=\bar{S}'_\alpha$. $\alpha\in W$, 其中 S_α 和 S'_α 是有限 C-代数而 W 是脚码的某一集合. 令 e_α 是代数 S_α 的单位元.

利用[11]中的结果,若 e 是任意幂等元,我们有

$$A=A_e(1)+A_e\left(\frac{1}{2}\right)+A_e(0),$$

$$A_e(1)A_e(0)=0, \tag{18}$$

其中 A 之子向量空间 $A_e(\lambda)$, $\lambda=1,\frac{1}{2},0$, 是由 A 中所有具性质 $xe=\lambda x$ 之元素 x 所组成的. 显然,有

$$S_\alpha\subseteq A_{e_\alpha}(1). \tag{19}$$

当 $\alpha\neq\beta$ 我们有 $A_{e_\alpha}(1)\subseteq A_{e_\beta}(0)$. 由此可见,

$$A_{e_\alpha}(1)\cap A_{e_\beta}(1)=0. \tag{20}$$

并知 $A_{e_\alpha+e_\beta}(1)\supseteq A_{e_\alpha}(1)+A_{e_\beta}(1)$. 故有

$$A_{e_\alpha+e_\beta+\cdots+e_\gamma}(1)\supseteq A_{e_\alpha}(1)+A_{e_\beta}(1)+\cdots+A_{e_\gamma}(1), \tag{21}$$

其中 $\alpha,\beta,\cdots,\gamma$ 是不相同的脚码.

因为等式 (20) 仍成立,假若把 e_α 和 e_β 看作是任意两个互相垂直的幂等元. 故, 由于 R 是有限的, 据 (21) 知, 包含有 R 中之元素的 $A_{e_\alpha}(1)$, $\alpha \in W$ 的个数只可能是有限的,说是 $A_{\alpha_i}(1), i = 1,2,\cdots,n$. 并且可认为 $A_e(1) \cap R = 0$,其中 $e = \sum_{j=1}^{m} e_{\beta_j}, \beta_j \neq \alpha_i, i = 1,2,\cdots,n$.

约定好,在下面将出现的脚码都是不同于 $\alpha_i, i = 1,2,\cdots,n$ 的.

令 $R_{e_\beta}(\lambda) = R \cap A_{e_\beta}(\lambda)$,由于 R 是理想子代数,

故 $R = R_{e_\beta}\left(\frac{1}{2}\right) + R_{e_\beta}(0)$,及 $R_{e_\beta}\left(\frac{1}{2}\right) \cap R_{e_\gamma}\left(\frac{1}{2}\right) = 0, \beta \neq \gamma$,这是因为 $R_{e_\beta}(1) = 0$ 而 $R_{e_\beta}\left(\frac{1}{2}\right) \cap R_{e_\gamma}\left(\frac{1}{2}\right) \subseteq R_{e_\beta+e_\gamma}(1) = 0$.

今证
$$R_{e_\beta+e_\gamma}(0) = R_{e_\beta}(0) \cap R_{e_\gamma}(0), \tag{22}$$
$$R_{e_\beta+e_\gamma}\left(\frac{1}{2}\right) = R_{e_\beta}(0) \cap R_{e_\gamma}\left(\frac{1}{2}\right) + R_{e_\beta}\left(\frac{1}{2}\right) \cap R_{e_\gamma}(0). \tag{23}$$

令 $x \in R$,易见,$e_\beta x = \frac{1}{2}x_{e_\beta}\left(\frac{1}{2}\right), e_{\gamma_x} = \frac{1}{2}x_{e_\gamma}\left(\frac{1}{2}\right)$①. 若 $x \in R_{e_\beta+e_\gamma}(0)$,则 $0 = (e_\beta + e_\gamma)x = e_\beta x + e_\gamma x = \frac{1}{2}x_{e_\beta}\left(\frac{1}{2}\right) + \frac{1}{2}x_{e_\gamma}\left(\frac{1}{2}\right)$,即 $x_{e_\beta}\left(\frac{1}{2}\right) = -x_{e_\gamma}\left(\frac{1}{2}\right)$, 因之 $x_{e_\beta}\left(\frac{1}{2}\right) = -x_{e_\gamma}\left(\frac{1}{2}\right) = 0$, 这是因为 $R_{e_\beta}\left(\frac{1}{2}\right) \cap R_{e_\gamma}\left(\frac{1}{2}\right) = 0$. 故 $x \in R_{e_\beta}(0) \cap R_{e_\gamma}(0)$,即 $R_{e_\beta+e_\gamma}(0) \subseteq R_{e_\beta}(0) \cap R_{e_\gamma}(0)$. 显然 $R_{e_\beta+e_\gamma}(0) \supseteq R_{e_\beta}(0) \cap R_{e_\gamma}(0)$,故等式 (22) 得证.

若 $x \in R_{e_\beta+e_\gamma}\left(\frac{1}{2}\right)$,则
$$(e_\beta + e_\gamma)x = \frac{1}{2}x = \frac{1}{2}x_{e_\beta}\left(\frac{1}{2}\right) + \frac{1}{2}x_{e_\beta}(0) = \frac{1}{2}x_{e_\gamma}\left(\frac{1}{2}\right) + \frac{1}{2}x_{e_\gamma}(0).$$
另一方面
$$(e_\beta + e_\gamma)x = \frac{1}{2}x_{e_\beta}\left(\frac{1}{2}\right) + \frac{1}{2}x_{e_\gamma}\left(\frac{1}{2}\right),$$
因此
$$x_{e_\beta}\left(\frac{1}{2}\right) = x_{e_\gamma}(0), \quad x_{e_\gamma}\left(\frac{1}{2}\right) = x_{e_\beta}(0),$$
即 $x = x_{e_\beta}\left(\frac{1}{2}\right) + x_{e_\beta}(0) \in R_{e_\beta}\left(\frac{1}{2}\right) \cap R_{e_\gamma}(0) + R_{e_\beta}(0) \cap R_{e_\gamma}\left(\frac{1}{2}\right)$.

---

① 若 x 是 A 的元素,用 $x_e(\lambda), \lambda = 1, \frac{1}{2}, 0$, 表 x 在 $A_e(\lambda)$ 中的分量.

易见 $R_{e_\beta}\left(\frac{1}{2}\right) \cap R_{e_\gamma}(0) + R_{e_\beta}(0) \cap R_{e_\gamma}\left(\frac{1}{2}\right) \subseteq R_{e_\beta+e_\gamma}\left(\frac{1}{2}\right)$. 这样等式(23)也得证.

值得注意的, 等式 (22) 和 (23) 对具下面性质的任意两互相垂直的幂等元 e_1, e_2 都是成立的: $R_{e_1}(1) = R_{e_2}(1) = R_{e_1+e_2}(1) = 0$. 据等式(22)及由于 R 是有限的, 必存在幂等元 $e = \sum_{i=1}^{m} e_{\beta_i}$, 使 $R_e(0)$ 是最小的, 即言, 对任意脚码 $\gamma \neq \beta_i, i = 1,2,\cdots,m, R_{e_\gamma}(0) \supseteq R_e(0)$, 由

$$\begin{aligned} R &= R_{e_\gamma+e}\left(\frac{1}{2}\right) + R_{e_\gamma+e}(0) \\ &= R_{e_\gamma}\left(\frac{1}{2}\right) \cap R_e(0) + R_{e_\gamma}(0) \cap R_e\left(\frac{1}{2}\right) + R_{e_\gamma}(0) \cap R_e(0) \\ &= R_{e_\gamma}(0) \cap R_e\left(\frac{1}{2}\right) + R_e(0) \\ &= R_e\left(\frac{1}{2}\right) + R_e(0), \end{aligned}$$

得 $R_{e_\gamma}(0) \supseteq R_e\left(\frac{1}{2}\right)$, 因之 $R_{e_\gamma}(0) \supseteq R_e\left(\frac{1}{2}\right) + R_e(0) = R$, 即当 $\gamma \neq \beta_i$ 时, 有 $R_{e_\gamma}\left(\frac{1}{2}\right) = 0$.

至此我们已得到下面事实: 除去 W 中有限个脚码之外, $A_{e_\gamma}(0) \supseteq R$. 因之, 据(18)(19), 得 $S_\alpha R = 0$. 故可以把 P 及 P' 写成下面形状

$$P = P_1 + P_2, \quad P' = P_1' + P_2',$$

其中 $\bar{P}_i = \bar{P}_i', i = 1,2$. P_1 是有限 C-代数而 $P_2 R = 0$. 重复一下我们在预理 9 之证明的末尾所作的讨论, 便得

$$P\sigma = P',$$

其中 σ 是 R-内自同构, 它的形状在 §4 中给出过.

定理 4　令 R 表若当代数 A (在特征数为 0 的域上) 之局部 N-根, 若 R 是有限的而 $\bar{A} = \frac{A}{R}$ 是任意多个维数不多于可数的局部 C-代数的直和, 则 (i) A 是可分解的, (ii) P' 和 P 在 A 中是 R-弱共轭.

据定理 2, 3 可以证明这一定理, 其办法就和在[1] 根据预理 2 来证明定理 1 时所用的一样.

§8. 预　　理

预理 12　设 R 是结合代数, 或交错代数, 或若当代数或李代数 A 的

理想子代数. 设$\bigcap_{n=1}^{\infty} R_n = 0$.($\bigcap_{n=1}^{\infty} R^{[n]} = 0$),且对 $n = 1,2,\cdots$ 有等式 $A = S_n + R$,其中 S_n 是子代数而$S_n \cap R = R_n(S_n \cap R = R^{[n]})$,且有 $S_{n+1} \subseteq S_n$. 若A 在由 $\{R_n\}(\{R^{[n]}\})$ 所定义的拓扑结构中是完全的,则令 $S = \bigcap_{n=1}^{\infty} S_n$,必有 $A = S + R$,显然还有 $S \cap R = 0$.

证明:若 a 是A 中任意一元素,则对任意 n 言,存在有 S_n 中之元素 s_n,使 $a \equiv s_n(\mathrm{mod}\ R)$. 因数 $s_{n+1} - s_n \in S_n \cap R = R_n(R^{[n]})$. 故 $\{s_n\}$ 是收敛叙列,由于A 是完全的,则 $\{s_n\}$ 收敛于A 之某一元素s. 因为S_n 是子代数而在所给定的拓扑结构中又是开集合,故 S_n 是封闭的,因之对所有的 n 言$s \in S_n$,即 $s \in \bigcap_{n=1}^{\infty} S_n = S$. 所以 $A = S + R$.

预理 13 令 R 是结合代数A 的理想子代数. 若$\bigcap_{n=1}^{\infty} R_n = 0$,而 A 在 R - 拓扑结构中是完全的. 则对任意给定的 $x \in R$ 存在一元素$x' \in R$,使 $xx' = x'x = x + x'$,而 $\sigma_x = 1 - R_x - L_{x'} + L_{x'}R_x$ 是A 的自同构,若 $y \in R$,则 $\sigma_x\sigma_y = \sigma_{x-xy+y}$.

证明:显然,$-\sum_{k=1}^{\infty} x^k$ 收敛于R 中某一元素x',且有 $xx' = x'x = x + x'$. 令 a 表示A 之任意元素我们有

$$
\begin{aligned}
a\sigma_x\sigma_y &= (a - ax - x'a + x'ax) - (ay - axy - x'ay + x'axy) - \\
&\quad (y'a - y'ax - y'x'a + y'x'ax) + \\
&\quad (y'ay - y'axy - y'x'ay + y'x'axy) \\
&= a - a(x - xy + y) - (x' - y'x' + y')a + \\
&\quad (x' - y'x' + y')a(x - xy + y) \\
&= a\sigma_{x-xy+y}.
\end{aligned}
$$

这是因为若令 $z = x - xy + y, z' = x' - y'x' + y$,易检验是有等式 $zz' = z'z = z + z'$ 的.

由上可得 $\sigma_x\sigma_{x'} = 1$ (恒等变换),而这等式说明 σ_x 是A 到A 上的一一对应. 易检验 $(ab)\sigma_x = (a\sigma_x)(b\sigma_x), a, b \in A$. 因此 σ_x 是A 的自同构.

预理 14 令 R 是交错代数,或若当代数或代数的李代数 A (在特征数为 0 的域上的) 的理想子代数. 若 $A = \frac{A}{R}$ 是局部有限的,若$\bigcap_{n=1}^{\infty} R_n =$

$0\left(\bigcap_{n=1}^{\infty} R^{[n]}=0\right)$，且 A 在 $\{R_n\}(\{R^{[n]}\})$ 所定义的 R-拓扑结构中是完全的，则 $\sigma=\exp D=1+D+\frac{D^2}{2!}+\cdots$，其中 D 之表示式在 §4 中给出，将是 A 的自同构.

证明：先证：$\sigma_n=\sigma(\bmod R_n(R^{[n]}))$①是代数 $\frac{A}{R_n}\left(\frac{A}{R^{[n]}}\right)$ 的自同构. 由预理 2 ~ 4 之推论知 $\frac{R}{R_n}\left(\frac{R}{R^{[n]}}\right)$ 是局部有限的，而据预理 2 ~ 4，由于 $\frac{A}{R}$ 及 $\frac{R}{R_n}\left(\frac{R}{R^{[n]}}\right)$ 是局部有限的，故 $\frac{A}{R_n}\left(\frac{A}{R^{[n]}}\right)$ 也是局部有限的. 因此，在 $\frac{A}{R_n}\left(\frac{A}{R^{[n]}}\right)$ 中任意有限个元素 $a,b,\cdots,c\in A$ 与在 D 之表示式中所出现的 x_i 及 y_i 在一起生成一个有限的子代数 $\bar{B}_{(n)}$，其 N-根包含元素 x_i，今有

a) 对交错代数和若当代数的情形. 由[6]，[7] 中的定理知 D 是有限代数 $B(n)$ 的幂零微分. 因此存在自然数 $m=m(a,n)$，使在代数 $\frac{A}{R_n}\left(\frac{A}{R^{[n]}}\right)$ 中有

$$aD^m=0,$$

即
$$aD^m\equiv 0\ (\bmod R_n(R^{[n]})). \tag{24}$$

b) 对李代数情形，显然，D 是微分和

$$aD^n\equiv 0\ (\bmod R_n=R^{[n]}). \tag{25}$$

由上及由[7] 中所谈到的事实：若 D 是在特征数为 0 之域上的代数 A 的幂零微分，则 $\exp D$ 是代数 A 的自同构，可知 σ 是子代数 $B_{(n)}\subseteq\frac{A}{R}\left(\frac{A}{R^{[n]}}\right)$ 的自同构，因之也是局部有限代数 $\frac{A}{R_n}\left(\frac{A}{R^{[n]}}\right)$ 的自同构，即

$$(ab)\sigma\equiv(a\sigma)(b\sigma)(\bmod R_n(R^{[n]})).$$

据(24)，(25) 并注意到 A 之完全性，可知对任意元素 $a\in A$ 元素 $a\exp D$ 是存在的.

因为对所有的 n 有 $(ab)\sigma\equiv(a\sigma)(b\sigma)(\bmod R_n(R^{[n]}))$，故 $(ab)\sigma=(a\sigma)(b\sigma)$. 若 $\sigma^{-1}=\exp(-D)$，则 $a\sigma^{-1}$ 存在，且对所有的 n 有 $a\sigma^{-1}\sigma\equiv$

① $\sigma_1\equiv\sigma_2(\bmod R_n)$ 是指对任意 A 之元素 a 有 $a\sigma_1\equiv a\sigma_2(\bmod R_n)$.

$a\sigma\sigma^{-1} \equiv a(\bmod R_n(R^{[n]}))$，故 $a\sigma^{-1}\sigma = a\sigma\sigma^{-1} = a$，即 σ 是 A 到 A 的一一对应，因此 σ 是 A 之自同构.

预理 15 令 R 表代数的李代数 A（在特征数为 0 之域上）之理想子代数. 设 $\frac{A}{R}$ 是局部有限的. 若 $\bigcap_{n=1}^{\infty} R_n = 0$ 而 A 在 R - 拓扑结构中是完全的.

设 $\sigma_x = \exp R_x \cdot \sigma_y = \exp R_y$，其中 $x \in R, y \in R_n$，

则 $\sigma_x\sigma_y \equiv \sigma_{x+y}(\bmod R_{n+1})$.

这可由 $R_xR_y \equiv R_yR_x \equiv R_yR_y \equiv 0(\bmod R_{n+1})$ 得出.

预理 16 设 R 是结合代数或交错代数或若当代数或代数的李代数 A（在被允许的特征数的域上）的理想子代数. $\frac{A}{R}$ 是局部有限的，设 $\bigcap_{n=1}^{\infty} R_n = 0\left(\bigcap_{n=1}^{\infty} R^{[n]} = 0\right)$，$A$ 在 R - 拓扑结构中是完全的. Q 和 Q' 是 A 的两个局部 C - 代数. 设 $Q \equiv Q'(\bmod R)$，并设对任意自然数 n 言，由 $Q_n \equiv Q'(\bmod R_n(R^{[n]}))$，其中 Q_n 是 A 中的局部 C - 代数，可推得存在 A 之一 $R_n(R^{[n]})$ - 内自同构 σ_{n+1}，使 $Q_n\sigma_{n+1} \equiv Q'(\bmod R_{n+1}(R^{[n+1]}))$，其中

a) 在结合代数的情形，$\sigma_{n+1} = \sigma_{x_{n+1}}, x_{n+1} \in R_n$；

b) 在交错代数的情形，$\sigma_{n+1} = \prod_{i=1}^{m_{n+1}} \exp D_{n+1,i}$，其中 m_{n+1} 是自然数，$D_{n+1,i} = \sum_{j=1}^{t_{n+1,i}} D_{y_{n+1,i,j}x_{n+1,i,j}}$，其中 $t_{n+1,i}$ 是自然数. $x_{n+1,i,j} \in R_n(R^{[n]})$；

c) 在若当代数的情形，$\sigma_{n+1} = \prod_{i=1}^{m_{n+1}} \exp D_{n+1,i}$，其中 m_{n+1} 是自然数，$D_{n+1,i} = \sum_{j=1}^{t_{n+1,i}} D_{y_{n+1,i,j}x_{n+1,i,j}}$，这里的 $t_{n+1,i}$ 是自然数而 $x_{n+1,i,j} \in R_n$；

d) 在李代数的情形，$\sigma_{n+1} = \sigma_{x_{n+1}} = \exp R_{x_{n+1}}$，其中 $x_{n+1} \in R_n$，

则那时必存在 A 的一个 R - 内自同构，使得 $Q\sigma = Q'$，并且 a) 在结合代数的情形 $\sigma = \sigma_x = 1 - R_x - L_{x'} + L_{x'}R_x$，其中 $x, x' \in R$ 而 $xx' = x'x = x + x'$；b) 在李代数的情形 $\sigma = \sigma_x = \exp R_x, x \in R$.

证明： a) 对结合代数和李代数的情形据预理 13，14 和本预理的假设，我们有

$$Q \equiv Q'(\bmod R_1);$$

$$Q\sigma_{x_2} \equiv Q'(\text{mod } R_2), x_2 \in R_1;$$

$$Q\sigma_{x_2}\sigma_{x_2} \equiv Q\sigma_{x_2+x_3} \equiv Q'(\text{mod } R_3), x_3 \in R_2;$$

$$\cdots\cdots\cdots\cdots$$

$$Q\sigma_{x_2+\cdots+x_n}Q\sigma_{x_{n+1}} \equiv Q\sigma_{x_2+\cdots+x_n+x_{n+1}} \equiv Q'(\text{mod } R_{n+1}), x_{n+1} \in R_n;$$

$$\cdots\cdots\cdots\cdots$$

因为级数 $\sum_{k=2}^{\infty} x_k$ 是收敛的而 $x_k \in R$，故 $\sum_{k=2}^{\infty} x_k = x \in R$. 易见，对任意 n 言，有

$$Q\sigma_x \equiv Q\sigma_{x_1+\cdots+x_n} \equiv Q'(\text{mod } R_n),$$

即 $Q\sigma_x = Q'$，其中在结合代数的情形 $\sigma_x = 1 - R_x - L_{x'} + L_{x'}R_x$，在李代数的情形 $\sigma_x = \exp R_x$. 且在两种情形 σ_x 都是A 之自同构，这个我们在预理 13，14 中证过了.

b) 对交错代数和若当代数的情形，据假设对任意 $n \geqslant 2$ 我们有

$$Q \equiv Q'(\text{mod } R_1),$$

$$Q\sigma_2\sigma_3\cdots\sigma_n \equiv Q'(\text{mod } R_n(R^{[n]})).$$

注意到预理中对 σ_{n+1} 之假设，易见，对任意 A 之元素 a 有 $a - a\sigma_{n+1} \equiv 0(\text{mod } R_n(R^{[n]}))$，故 $a\sigma_2\cdots\sigma_n - a\sigma_2\cdots\sigma_n\sigma_{n+1} \equiv 0(\text{mod } R_n(R^{[n]}))$. 因此叙列 $\{a\sigma_2\cdots\sigma_n\}$ 是收敛的，由于A 是完全的，这叙列收敛于A 中之某一元素 b，我们把它写成 $b = a\sigma$，其中 $\sigma = \sum_{i=2}^{\infty}\sigma_i$. 因为 $\sigma_2\cdots\sigma_n$ 是A 的自同构，则$(\sigma_1\sigma_2)\sigma = (a_1\sigma)(a_2\sigma)$. 至此只剩下来要证 σ 是A 到A 的一一对应，现在来证它. 令 $\sigma_{n+1}^{-1} = \prod_{i=m_{n+1}}^{1} \exp(-D_{n+1,i})$，易见 σ_{n+1}^{-1} 是 σ_{n+1} 的逆自同构，而叙列 $\{a\sigma_n^{-1}\cdots\sigma_2^{-1}\}$ 收敛到 A 之某一元素，写成 $a\sigma^{-1}$，其中 $\sigma^{-1} = \cdots\sigma_n^{-1}\sigma_{n-1}^{-1}\cdots\sigma_2^{-1}$. 注意到预理中对 σ_{n+1} 之假设，易见对任意 n 有 $a = (a\sigma^{-1})(\sigma_2\sigma_3\cdots\sigma_n) \equiv 0(\text{mod } R_n(R^{[n]}))$，因此 $a - (a\sigma^{-1})\cdot\sigma = 0$，即 $(a\sigma^{-1})\sigma = a$. 这样证明了 σ 是A 到A 的一一对应，显然 $Q\sigma = Q'$.

§9. 在 R－拓扑结构中完全的代数的分解

定理5　令 R 是结合代数，或交错代数，或若当代数或代数的李代数

A(在被允许的特征数的域上)的理想子代数. 设$\bigcap_{n=1}^{\infty} R_n = 0$, 若$A$在$R$-拓扑结构中是完全的, 且$\frac{A}{R}$是维数不多于可数的局部$C$-代数, 则$A$是可分解的.

这定理的证明完全是和预理 7 的证明相平行的. 设对某一自然数k有$A = S_k + R$, 其中S_k是子代数, $S_k \cap R = R_k$(当$k = 1$时, 显然我们有这样的等式). 现在来研究代数$\frac{S_k}{R_{k+1}}$. 有

$$\left(\frac{R_k}{R_{k+1}}\right)^3 = 0 \quad 及 \quad \frac{\frac{S_k}{R_{k+1}}}{\frac{R_k}{R_{k+1}}} \simeq \frac{A}{R},$$

由是得$\frac{R_k}{R_{k+1}}$是代数$\frac{S_k}{R_{k+1}}$的局部N-根. 由定理 1 得

$$\frac{S_k}{R_{k+1}} = \frac{S_{k+1}}{R_{k+1}} + \frac{R_k}{R_{k+1}},$$

即$S_k = S_{k+1} + R_k$, 其中S_{k+1}是子代数$\frac{S_{k+1}}{R_{k+1}}$在A中的完全原像而$S_{k+1} \cap R_k = R_{k+1}$. 故得$A = S_{k+1} + R$, 其中$S_{k+1} \cap R = S_{k+1} \cap S_k \cap R = R_{k+1}$且$S_{k+1} \subseteq S_k$. 令$S = \bigcap_{n=1}^{\infty} S_n$. 由预理 12 得$A = S + R$, 其中$S \cap R = 0$.

定理 6 若A, R及$\frac{A}{R}$就像定理 5 中所说的那样, 则Q和P的某一子代数Q'是R-弱共轭的, 即对Q之任意有限C-子代数有$Q_1\sigma \subseteq Q'$, 其中σ是R-内自同构, 并且尚能断定, 对结合代数的情形$\sigma = 1 - R_x - L_{x'} + L_{x'}R_x$, $x \in R$. 对李代数的情形$\sigma = \exp R_x$, $x \in R$.

证明: 易见, $Q \cap R \equiv 0(\mathrm{mod} R_n)$, 故$Q \cap R = 0$. 子代数$\overline{Q}_1$中的陪集在$P$内的代表(当然, Q_1中之每一陪集在P内有一个且仅有一个代表)组成P之一子代数. 命之为Q_1'. 显然$Q_1 \equiv Q_1'(\mathrm{mod}\ R)$.

由预理 16, 为了证明定理, 只需证明: 若$Q_n \equiv Q_1'(\mathrm{mod} R_n)$, 则存在$A$之$R_n$-自同构$\sigma_{n+1}$, 使$Q_n\sigma_{n+1} \equiv Q'(\mathrm{mod} R_{n+1})$, 并且$\sigma_{n+1}$要具有在预理 16 中所写出的形状, 我们来证这一点.

来研究代数$A_n = Q_n + R_n = Q_1' + R_n$. 取其商代数$\frac{A_n}{R_{n+1}}$, 则有

$$\frac{\frac{A_n}{R_{n+1}}}{\frac{R_n}{R_{n+1}}} \simeq \frac{A_n}{R_n} \simeq Q_1',$$

并且 $\left(\frac{R_n}{R_{n+1}}\right)^3 = 0$,而对李代数的情形有

$$\left(\frac{R_n}{R_{n+1}}\right)^2 = 0.$$

由定理 1 和预理 10 知,存在代数 $\frac{A_n}{R_{n+1}}$ 的 $\frac{R_n}{R_{n+1}}$ - 内自同构-σ_{n+1},使在代数 $\frac{A_n}{R_{n+1}}$ 中有等式

$$Q_n\sigma_{n+1} = Q_1',$$

而 $\sigma_{n+1} = 1 + f(R(x_i)_n; L(y_j)_n)$,其中 $(x_i)_n$ 及 $(y_j)_n$ 属于 $\frac{A_n}{R_{n+1}}$,并且 σ_{n+1} 的正确表示式就是具有在预理16 中所写出的那样. 据预理 13,14 知,若在陪集 $(x_i)_n$,$(y_j)_n$ 中顺序任取代表,说是 x_i,y_j,则表示式 $1 + f(R_{x_j}; L_{y_j})$ 将是 A 的 R_n - 内自同构. 仍用 σ_{n+1} 表自同构 $1 + f(R_{x_j}; L_{y_j})$,得

$$Q_n\dot{\sigma}_{n+1} \equiv Q_1' (\mathrm{mod}\ R_{n+1}).$$

§ 10. 在 $R^{[n]}$ - 拓扑结构中的代数的分解

定理 7　令 R 是结合代数,或交错代数或代数的李代数 A(在被允许的特征数的域上) 的理想子代数. 设 $\bigcap_{n=1}^{\infty} R^{[n]} = 0$. 叙列 $\{R^{[n]}\}$ 赋予 A 及 R 以拓扑结构,若在此拓扑结构中 R 是线性紧致 A - 模. 而 $\frac{A}{R}$ 是任意多个有限 C - 代数的直和,则(i) A 是可分解的,(ii) P' 和 P 在 A 中是 R - 共轭的. 即 $P\sigma = P'$,其中 σ 是 R - 内自同构,并且对结合代数言 $\sigma = 1 - R_x - L_{x'} + L_{x'}R_x$, $x, x' \in R$. 对李代数言 $\sigma = \exp R_x$, $x \in R$.

这定理的证明和定理 5,6 的证明完全相仿,只要利用到预理 7,9,11,12,13,14,16 以及下面的预理 17.

预理 17　若 A, R 就像在定理 7 中所说的那样,则 A 在 $R^{[n]}$ - 拓扑结构中是完全的.

证明：令 $\sum_{n=0}^{\infty} a_n$ 是收敛级数，并可认为 $a_n \in R^{[n]}$，$n \geqslant 1$. 显然 A－模 R 中的线性集合 $\sum_{i=1}^{k} a_n + R^{[k+1]}$ 组成有有限交性质的集合 E. 由于 A－模 R 是线性紧致的. 故 E 有不空之交，若 x 是这个交的任意一元素，则易见

$$\sum_{n=0}^{\infty} a_n = a_0 + x,$$

这样证明了 A 在 $R^{[n]}$ 拓扑结构中是完全的.

Курочкин 在 [12] 中所给出的例子说明. 在定理 7 中不能把 R 之线性紧致的条件取消.

由定理 7 和预理 9 立得下面定理.

定理 8　令 R 是结合代数或交错代数或代数的李代数 A（在被允许的特征数的域上）的理想子代数，设存在一自然数 n，使 $R^{[n]} = 0$，并设对于包含在 R 中的 A 之理想子代数最小条件是成立的. 若 $\frac{A}{R}$ 是任意多个有限 C－代数的直和，则 (i) A 是可分解的，(ii) P' 和 P 在 A 中 R－共轭.

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关于一种有限非结合代数

On a Class of Finite-Dimensional Non-Associative Algebras

我们知道对于有限结合代数、有限交错代数、有限若当代数和有限李代数有许多相平行的概念,例如根的概念,及许多相平行的定理,例如下面的定理:半单纯代数是单纯代数的直和;有限个单纯代数的直和是半单纯代数.能否将这些相平行的概念统一在一个概念里,将这些相平行的事实统一在一个定理里,是个有趣的问题.A.Узков 在[1]中将对于结合代数和李代数的上述定理糅合在一起,而 A.A.Albert 在[2]中将对于结合代数、若当代数、交错代数的上述定理糅合在一起.本文的目的在于把 Узков 和 Albert 的结果适当地结合起来,即:定义一种有限非结合代数—— 将称之为 λ - 对称代数.它包含交错代数(当然更包含结合代数了)、若当代数和李代数.对 λ - 对称代数定义根、半单纯代数及单纯代数等概念,而对上述四种代数言,这些概念与它们原来所具有的关于根、半单纯代数和单纯代数等概念相吻合.我们证明对 λ - 对称代数的上述定理.因而把开头提到的相平行的事实统一在一起.

现在我开始定义 λ - 对称代数.

令 A 表域 F(其特征数为零)上的有限代数(即对其乘法,除去它对加法的分配律外,不加任何限制).用 R_x, L_x 顺序表在 A 中用元素 $x \in A$ 所作的右、左乘积(即 $aR_x = ax$, $aL_x = xa$,其中 $a \in A$).显然 R_x, L_x,因而其乘积,都是 A 的线性变换.对 A 的任一基底言,A 的线性变换 T 是和一 n 级矩阵 T^* 相对应,其中 n 是有限代数 A 的维数.我们把矩阵 T^* 的迹叫做线性变换 T 的迹,将用 $Tr(T)$ 表之.我们知道 $Tr(T)$ 是和 A 之基底

的选择无关.

若 $x,y\in A$,设

$$\varphi(x,y)=\lambda_1 Tr(R_{xy})+\lambda_2 Tr(L_{xy})+\lambda_3 Tr(R_xR_y)+\lambda_4 Tr(R_xL_y)$$
$$+\lambda_5 Tr(L_xR_y)+\lambda_6 Tr(L_xL_y)$$

其中 $\lambda\equiv(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)$ 是从 F 中任意取定的一组值.我们管 $\varphi(x,y)$——这个以 $A\times A$ 为变域,函数值在 F 中的函数——叫做 A 的 λ-迹函数.易见 $\varphi(x,y)$ 是 x,y 的线性函数.

我们叫 A 是 λ-对称代数(这是 Узков 所用的名称),若其 λ-迹函数 $\varphi(x,y)$ 满足下列条件:

Ⅰ.对任意 $x,y\in A$ 有:$\varphi(x,y)=\varphi(y,x)$;

Ⅱ.对任意 $x,y,z\in A$ 有:$\varphi(xy,z)=\varphi(x,yz)$;

Ⅲ.A 的同态像的 λ-迹函数也满足上述条件 Ⅰ,Ⅱ.

由 Ⅰ,Ⅱ 易得 $\varphi(x,yz)=\varphi(zx,y)$.

设 $e_1,\cdots,e_n$ 组成 A 的基底.若设 $a_{ij}=\varphi(e_i,e_j)$,$x=\sum\limits_i\xi_ie_i$,$y=\sum\limits_i\eta_je_j$,易见 $\varphi(x,y)=\sum\limits_{i,j}\xi_i\eta_ja_{ij}$.当变换 A 的基底时,矩阵 $|a_{ij}|$ 变成矩阵 $M|a_{ij}|M'$,其中 M 是联系新旧基底的,因而是非退化矩阵.$\varphi(x,y)$ 叫做非退化的,若对 A 的某一基底(说是 $e_1,\cdots,e_n$)言,$\varphi(x,y)$ 所对应的矩阵 $|a_{ij}|$ 是非退化的.由上述显见,$\varphi(x,y)$ 之非退化与否由 A 及 λ 决定,与基底的选择无关.当 $\varphi(x,y)$ 满足 Ⅰ 时,不难看到 $\varphi(x,y)$ 是非退化的充要条件是在 A 中不存在元素 $x\neq0$,使对 A 之任意元素 a 都有 $\varphi(x,a)=0$ (将简写成 $\varphi(x,A)=0$).这一点在下面将经常用到.

我们现在来定义 λ-对称代数 A 的根.设 R_0 是 A 中所有具性质 $\varphi(x,A)=0$ 之元素 x 所组成的集合.因为

1.$\varphi(x,y)$ 是双线性的,故若 $\varphi(x,A)=\varphi(y,A)=0$,则 $\varphi(\alpha x+\beta y,A)=\alpha\varphi(x,A)+\beta\varphi(y,A)=0$,其中 α,β 为 F 中任意元素.因之 R_0 是 A 的子空间;

2.由条件 Ⅰ,Ⅱ 知:若 $\varphi(x,A)=0$ 而 $a\in A$,则 $\varphi(xa,A)=\varphi(x,aA)=0$,$\varphi(ax,A)=\varphi(x,Aa)=0$.

故 R_0 是 A 的理想子代数(由上面的证明亦得出:若 B 是 A 的任意理想子代数,则 A 中所有具性质 $\varphi(x,B)=0$ 之元素的集合是 A 的理想子代数.

这一事实以后将用到). 今考察商代数 $A/R_0 = A_1$. 由条件 Ⅲ 知 A_1 之 λ - 迹函数 $\varphi_1(x,y)$ 满足条件 Ⅰ, Ⅱ. 设 R_1 是 A_1 中所有具性质 $\varphi_1(x,A_1) = 0$ 之元素 x 的集合, 则 R_1 是 A_1 的理想代数. 设商代数 $A_1/R_1 = A_2$. 这样继续作下去, 我们将达到代数 A_n, 而 R_n 已经是零理想子代数了. 显然 A 和 A_n 是同态的. 设这个同态对应的核是 R, 则 $A/R \simeq A_n$. 易见这里的 A_n 有两种可能: 或者 A_n 本身是零代数, 即 $A = R$, 否则 A_n 的 λ - 迹函数是非退化的.

就称这样得来的 A 之理想子代数 R 为 A 之根. 根为零理想子代数的非零 λ - 对称代数称为半单纯代数. 没有真正理想子代数的 λ - 对称代数, 若其 λ - 迹函数 $\varphi(x,y) \not\equiv 0$ (即当 $\varphi(x,y)$ 非退化时), 则称之为单纯代数.

由上述立得

定理 1 λ - 对称代数 A 是半单纯的, 当且仅当其 λ - 迹函数是非退化的.

定理 2 若 R 是 λ - 对称代数 A 之根, 且 $R \neq A$, 则商代数 A/R 是半单纯代数.

现在来证下面的定理,

定理 3 λ - 对称代数 A 是半单纯的, 当且仅当 A 是有限个单纯代数的直和.

先证下面几个预理,

预理 1 若 λ - 对称代数 A 是半单纯的, 则在 A 中不存在非零的理想代数 B, 有性质 $B^2 = 0$.

证明: 设 $\varphi(x,y)$ 为 A 之 λ - 迹函数, 则由定理 1, $\varphi(x,y)$ 是非退化的. 假设如预理中所描述的理想子代数 B 存在. 设 $e_1,\cdots,e_r$ 是 B 的基底, 而 $e_1,\cdots,e_r,e_{r+1},\cdots,e_n$ 是 A 的基底, 其中 $r \geqslant 1$. 为了证明预理只需证: 对于这一基底言, 对任意 i 说, 矩阵 $R^*_{e_1e_i}, L^*_{e_1e_i}, R^*_{e_1}R^*_{e_i}, R^*_{e_1}L^*_{e_i}, L^*_{e_1}R^*_{e_i}, L^*_{e_1}L^*_{e_i}$ 等之迹都等于零, 因为那时将有 $\varphi(e_1,A) = 0$, 因之 $\varphi(x,y)$ 不能是非退化的了, 而得出矛盾.

显然当 $b \in B$ 时, R^*_b 及 L^*_b 因而 $R^*_{e_1}$ 及 $L^*_{e_1}$, 都有下面矩阵 M_1 的形状:

$$M_1 = \left|\begin{array}{c|c} 0 & 0 \\ \hline 0 & * \end{array}\right| \begin{array}{l} \}r \quad 列 \\ \}h-r \ 列 \end{array}$$

故立得 $Tr(R_{e_1e_i}) = Tr(L_{e_1e_i}) = 0$，其中 $1 \leqslant i \leqslant n$. 另一方面当 $1 \leqslant i \leqslant n$ 时，矩阵 $R_{e_i}^*$ 及 $L_{e_i}^*$ 有下面矩阵 M_2 的形状：

$$M_2 = \left|\begin{array}{c|c} * & 0 \\ \hline * & * \end{array}\right| \begin{array}{l} \}r \quad 列 \\ \}h-r \ 列 \end{array}$$

故矩阵 $R_{e_1}^*R_{e_i}^*, R_{e_1}^*L_{e_i}^*, L_{e_1}^*R_{e_i}^*$ 及 $L_{e_1}^*L_{e_i}^*$ 等必有矩阵 M_1M_2 的形状，即具下面矩阵之形状：

$$M_1M_2 = \left|\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array}\right| \begin{array}{l} \}r \quad 列 \\ \}h-r \ 列 \end{array}$$

故 $Tr(R_{e_1}R_{e_i}) = \cdots = Tr(L_{e_1}L_{e_i}) = 0$.

预理 2　若 A 是 λ - 对称代数，A' 是 A 之理想子代数而 $\varphi(x,y)$ 及 $\varphi'(x,y)$ 顺序为 A、A' 的 λ - 迹函数，则对 $x, y \in A'$ 言有 $\varphi(x,y) = \varphi'(x,y)$，因而 $\varphi'(x,y)$ 也满足条件 Ⅰ，Ⅱ.

证明： 设 $e_1, \cdots, e_r$ 是 A' 的基底而 $e_1, \cdots, e_r, e_{r+1}, \cdots, e_n$ 是 A 的基底. 若 $x, y \in A'$，对上述基底言有

$$R_x^* = \left(\begin{array}{c|c} R_x'^* & 0 \\ \hline * & 0 \end{array}\right), L_y^* = \left(\begin{array}{c|c} L_y'^* & 0 \\ \hline * & 0 \end{array}\right),$$

$$R_x^*L_y^* = \left(\begin{array}{c|c} R_x'^*L_y'^* & 0 \\ \hline * & 0 \end{array}\right), \cdots,$$

其中 $R_x'^*, L_y'^*$ 是 x, y 顺序在 A' 中所作的右乘积 R_x'，左乘积 L_y' 对上述 A' 之基底所对应的矩阵. 由之立得 $Tr(R_x) = Tr(R_x'), Tr(L_y) = Tr(L_y'), Tr(R_xL_y) = Tr(R_x'L_y'), \cdots$

预理 3　λ - 对称代数 A 的直和项 A' 本身也是 λ - 对称代数.

由于 A' 之同态像必是 A 的同态像及预理 2 立得上预理.

现在来证定理 3.

证明：（ⅰ）设 A 是半单纯代数，由定理 1，A 之 λ - 迹函数 $\varphi(x,y)$ 是非退化的. 在 A 中任取一单纯理想子代数 A_1（即不存在 A 之非零理想子代数 C，有 $A_1 \supset C$）. 设 A_1' 是 A 中所有具性质 $\varphi(x,A) = 0$ 之元素 x 所组成的集合，由先前证过的事实知 A_1' 是理想子代数，命 $B = A_1 \cap A_1'$. 有两种可能：

1. $B = A_1$. 此时我们有 $\varphi(B,B) = 0$. 由预理 1 知 $B^2 \neq 0$. 故单纯理想子代数 B 可视为由某一非零元素 $b_1 b_2$，其中 $b_1, b_2 \in B$，在 A 中所生成的理想子代数. 即 B 中每一元素 b 可表为有下面形状的元素之和：$(b_1 b_2)T$，其中 T 是 A 中某些元素所作的右乘积或左乘积之积.

今证 $\varphi(A,B) = 0$. 为此只需证 $\varphi(A,(b_1 b_2)T) = 0$. 因为 $\varphi(B,B) = 0$，故 $\varphi(A, b_1 b_2) = \varphi(Ab_1, b_2) = 0$. 设 $\varphi(A,(b_1 b_2)T) = 0$，则对任意 $a \in A$ 有：

$$\varphi(A,(b_1 b_1)TR_a) = \varphi(aA,(b_1 b_2)T) = 0,$$

$$\varphi(A,(b_1 b_2)TL_a) = \varphi(Aa,(b_1 b_2)T) = 0,$$

由归纳法知对任意 T 言都有 $\varphi(A,(b_1 b_2)T) = 0$，即证得 $\varphi(A,B) = 0$. 而这是不可能的，因为 A 是半单纯的. 故 $B = A_1$ 是不可能的.

2. $B = 0$. 由于 $\varphi(x,y)$ 是非退化的，故若 A_1 的维数是 r，则 A_1' 之维数必等于 $n - r$. 因之 $A = A_1 + A_1'$（直和）. 由预理 3 知 A_1 及 A_1' 是 λ - 对称代数. 易见 A_1 是单纯代数而 A_1' 是半单纯代数. 用归纳法可得 A 是单纯代数的直和.

(ⅱ) 设 A 是有限个单纯代数的直和，即 $A = A_1 + \cdots + \dot{A}_m$，$A_i$ 是单纯代数. 设 A 及 $A_i (i = 1, 2, \cdots, m)$ 的 λ - 对称函数顺序为 $\varphi(x,y)$ 及 $\varphi_i(x,y)$，而 A 之元素 $x = \sum_i x_i$，$y = \sum_i y_i$，$x_i, y_i \in A_i$，$i = 1, 2, \cdots, m$，则易见

$$\varphi(x,y) = \sum_i \varphi(x_i, y_i) = \sum_i \varphi_i(x_i, y_i)$$

由于 $\varphi_i(x,y)$ 满足条件 Ⅰ, Ⅱ 故 $\varphi(x,y)$ 也满足 Ⅰ, Ⅱ. 另一方面，A 之同态像也必是若干个 A_i 的直和，因之其 λ - 迹函数也满足条件 Ⅰ, Ⅱ. 因此 A 是 λ - 对称代数. 今证 $\varphi(x,y)$ 是非退化的. 若不是，则必存在 $0 \neq \sum x_i = x \in A$，有 $\varphi(x,A) = 0$. 易见有某一 x_i，说是 x，是异于零的元素.

$$0 = \varphi(x, A_1) = \sum \varphi(x_i, A_1) = \varphi(x_1, A_1) = \varphi_1(x_1, A_1),$$

此与 A_1 是单纯代数相矛盾.

由预理 2，可得 λ - 对称代数 A 的根 $R \neq 0$ 具下面性质：存在 A 的理想子代数列 $R = R_n \supset R_{n-1} \supset \cdots \supset R_0 \supset R_{-1} = 0$ 有性质：R_k / R_{k-1}，$k = n, n-1, \cdots, 0$，其 λ - 迹函数恒等于零.

易见，结合代数和李代数是(0,0,1,0,0,0)-对称代数（可参看[1]）.由[3]知若当代数是(1,0,0,0,0,0)-对称代数，而由[4]知交错代数是(0,1,0,0,0,0)-对称代数.至于上面定义的关于λ-对称代数的根、半单纯代数、单纯代数等概念与这四种代数原来所具的相吻合，除去注意到刚才提到的根的性质外，关于结合代数的可参看[5]，关于若当代数的参看[3]，关于交错代数的可参看[4][6]，而关于李代数的则由 Cartan 的关于可解性判断定理和下面简单事实而得：可解李代数借助于可解李代数所得到的扩张仍是可解的.这样我们达到了本文开始所提出的目的.

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关于多元算子群中的直因子

On Direct Factors of Group with Multi-Operators

在[1]中引入了多元算子群的概念,我们把具有多元算子系Ω的多元算子群记为Ω-群.依[2]中引用过的符号,我们用下列字母表示Ω-群可以具有的一些性质:

ND:每一理想都是直因子;

SD:每一Ω-子群是半直因子;

R:每一Ω-子群是正则因子;

D:每一Ω-子群是直因子;

F:每一Ω-子群是自由因子;

它们的详细定义将在下面§1和§5中给出.一个具有性质P的代数系统将简记作P-代数系统.这样,在[2]讨论了P-群,其中$P = ND, SD, R, F$,在[3]中讨论了ND-带算子的群,在[5]中讨论了D-群,在[4]中讨论了D-环.本文中将刻画P-Ω-群,其中P可为上述每一种性质.我们知道群和环,以及带算子的群和环都是Ω-群的特殊情形,因而上面提到的关于群或环的已知结果都可以看作是本文结果的直接推论.

在§1中简单介绍一下Ω-群,§2 ~ 5给出上述五类Ω-群的结构定理,§6中给出本文结果的一些推论.

§1. 有关Ω-群的一些定义和定理

为了方便,在这里我们把Ω-群的定义及其一些性质简单提一下(参看原著[1]或[7]中之介绍).

设 G 是一个群，它不一定是 $Abel$ 群，但仍把其运算记成加法，其恒等元记成 0. $G\times G\times\cdots\times G$($n$ 个, $n\geqslant 1$) 到 G 内的对应 ω 叫做 G 上的一个 n 元运算，对元素组 $x_1,x_2,\cdots,x_n\in G$ 施行运算 ω 之结果记成 $x_1x_2\cdots x_n\omega$. 设 Ω 为 G 的一些多元运算的集，说 G 是带 Ω 的多元算子群，而记成 Ω-群，若对任意 n 元运算 $\omega\in\Omega$，有

$$\underbrace{00\cdots00}_{n\text{个}}\omega = 0$$

易见群，环以及有算子的群，环等等都可看成是 Ω-群，只要适当地去选择 Ω.

和通常一样定义 Ω-子群，同构，同态等概念.

设 G 是 Ω-群而 A 是 G 的子集，说 A 是 G 之理想，若

1. A 是加群 G 的正规子群

2. 对任意 n 元运算 $\omega\in\Omega$，任意元素 $a\in A$，任意元素 $x_1,\cdots,x_n\in G$ 以及任意 $i=1,2,\cdots,n$，有

$$-(x_1x_2\cdots x_n\omega)+x_1\cdots x_{i-1}(a+x_i)x_{i+1}\cdots x_n\omega\in A.$$

由之还知对理想 A 中任意元素 $a_1,\cdots a_n$ 还有

$$-(x_1\cdots x_n\omega)+(a_1+x_1)(a_2+x_2)\cdots(a_n+x_n)\omega\in A.$$

因而知理想 A 必是 Ω-子群,

和通常一样定义单 Ω-群.

设 A,B 是 Ω-群 G 的两个 Ω-子群，A,B 之相互换位子 $[A,B]$，依定义，就是在 A,B 生长的 Ω-子群 $\{A,B\}$(以下将使用此符号) 中由下面两类元素所生成的理想:

$$[a,b]=-a-b+a+b,\ a\in A,\ b\in B$$

以及

$$[a_1,a_2,\cdots,a_n;b_1,b_2,\cdots,b_n;\omega]=-a_1a_2\cdots a_n\omega-b_1b_2\cdots b_n\omega+(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)\omega,$$

其中 n 元运算 $\omega\in\Omega$ 而 $a_1,\cdots,a_n\in A$, $b_1,\cdots,b_n\in B$.

Ω-群 G 叫做 $Abel$-Ω-群，若 $[G,G]=0$.

Ω-群 G 之 Ω-子群 A 是理想当且仅当 $[A,G]\subseteq A$.

说 Ω-群 G 是它的 Ω-子群 $G_\alpha(\alpha\in\mathrm{I})$ 的直和，若 i) 对任意 $\alpha\in\mathrm{I}$，$[G_\alpha,\bar{G}_\alpha]=0$，其中 $\bar{G}_\alpha=\{G_\beta,\beta\neq\alpha,\beta\in\mathrm{I}\}$; ii) G 的每一非零元素 a 可

唯一的(不计元素次序)表成从有限个 G_α 中各取一个非零元素所作成的和.

上面的直和定义与下面的定义等价:说Ω-群 G 是它的Ω-子群 G_α, $\alpha \in I$ 的直和,若对任一 $\alpha \in I$, i) G_α 是 G 之理想;ii) $G_\alpha \cap \bar{G}_\alpha = 0$.

此时称 G_α 为 G 之直因子.

易见 G 之直因子是 G 之理想,在Ω-群 G 的某一直和中不同直因子间依元素可交换.

至此对于Ω-群言性质 D 和 ND 就完全清楚了.

说Ω-群 G 之Ω-子群 A 是 G 的正则因子(依照[8]),若存在一Ω-子群 B,使 $G = \{A, B\}$ 且若把 A 所生成的 G 的理想记作 (A)(以下将使用此符号)则还有

$$(A) \cap B = 0, A \cap (B) = 0$$

说Ω-群 G 之Ω-子群 A 是 G 之半直因子,若存在Ω-群 G 的一个理想 B,使 $G = A \dot{+} B$ 而 $A \cap B = 0$.

这样对于Ω-群言性质 R 和 SD 就完全清楚了.

约定把具有性质 M 的Ω-群记成 M-Ω-群而一切 M-Ω-群的集记成 $[M]$,则我们有显然的

定理 1 $[ND] \supseteq [SD] \supseteq [R] \supseteq [D]$

§2. ND-Ω-群

预理 1 ND-Ω-群 G 的每一理想 A 也是 ND-Ω-群.

证明: 首先有 $G = A \oplus B$. 设 C 是 A 之理想.
为了证明预理只要知道 C 是 G 之理想就够了. 对任意 $c \in C, g \in G$ 有 $[c, g] \in C$, 即 C 是群 G 之正规子群. 另一方面,若 $\omega \in \Omega, c_i \in C, g_i \in G, i = 1, 2, \cdots, n$, 则 $g_i = a_i + b_i, a_i \in A, b_i \in B$ 而有

$$\begin{aligned}
& - g_1 \cdots g_n\omega + (C_1 + g_1) \cdots (c_n + g_n)\omega \\
&= - (a_1 + b_1) \cdots (a_n + b_n)\omega + (c_1 + a_1 + b_1) \cdots (c_n + a_n + b_n)\omega \\
&= - a_1 \cdots a_n\omega - b_1 \cdots b_n\omega + b_1 \cdots b_n\omega + (c_1 + a_1) \cdots (c_n + a_n)\omega \\
&= - a_1 \cdots a_n\omega + (c_1 + a_1) \cdots (c_n + a_n)\omega \in C,
\end{aligned}$$

这里我们用到了 $[A, B] = 0$ 而 C 是 A 之理想. 因而 C 是 G 之理想,故 $G = C \oplus C'$ 而 $A = C \oplus A \cap C'$.

定理 2 Ω-群 G 是 ND-Ω-群当且仅当 Ω-群 G 是单 Ω-群 G_α, $\alpha \in \mathrm{I}$, 的直和.

证明: 设 G 是单 Ω-群 $G_\alpha, \alpha \in \mathrm{I}$, 的直和而 A 是 G 的一个理想, 显然

$$G = A + \sum_{\alpha \in \mathrm{I}} G_\alpha \tag{1}$$

将集 I 良序之. 若对 $\beta < \mu$, 这里 μ 是 I 中某一固定元素, 所有 G_β 之取舍已经确定, 而用 M 表示保留下来的 G_β 之足码集, 今确定 G_μ 之取舍. 若

$$\Big(A + \sum_{\beta \in M} G_\beta\Big) \cap G_\mu = 0,$$

则保留 G_μ, 否则由于 G_μ 之单性必

$$\Big(A + \sum_{\beta \in M} G_\beta\Big) \cap G_\mu = G_\mu,$$

此时把 G_μ 舍去. 依超限归纳法每一 G_α 之取舍完全确定, 设保留下来的 G_α 之足码集为 I′, 易证

$$G = A \oplus \sum_{\alpha \in \mathrm{I}'} \oplus G_\alpha,$$

即 A 是 G 之直因子, 因而 G 是 ND-Ω 群.

反之, 若 G 是 ND-Ω 群, 今证 G 是单 Ω-群之直和. 设 $G_\alpha, \alpha \in \mathrm{I}$, 是 G 之所有单纯理想(即 G 之非零最小理想) 的全体, 令 $G' = \{G_\alpha, \alpha \in \mathrm{I}\}$, 易见 G' 是 G 之理想, 因而

$$G = G' \oplus G'',$$

此时 G'' 必不含 G 之单纯理想. 设 $G'' \neq 0$, 而 a 是它的一个非零元素. 今考虑 a 在 G 中所生成的理想 $A \subseteq G''$. A 当然也不含 G 的单纯理想, 因而对 A 之任意理想 $B \subset A$ 必有理想 B' 使 $B \subset B' \subset A$, 否则 $A = B \oplus B''$ 而 B'' 是单纯理想. 故存在无限列.

$A_1 \subset A_2 \subset \cdots \subset A_n \subset \cdots \subset A$, 每一 A_i 是 A 的(因而也是 G 的) 理想, 依预理 1, A_i 及 $C = \bigcup_{i=1}^{\infty} A_i$ 是 ND-Ω-群, 故有非零理想 $B_i, i = 0, 1, 2, \cdots$ 以及 C' 使(设 $A_\circ = 0$)

$$A_{i+1} = B_{i+1} \oplus A_i,\ A = C \oplus C',$$

易见

$$A = \sum_{i=1}^{\infty} \oplus B_i \oplus C'. \tag{2}$$

另一方面 $a \in A$

$a = b_1 + \cdots + b_m + c \quad b_i \in B_i, i = 1,2,\cdots,m, c \in C'$ 而理想A 是由 a 生成的,故

$$A \subseteq B_1 + \cdots + B_m + C',$$

这与(2) 矛盾,因而 $G'' = 0$,即 $G = G' = \{G_\alpha, \alpha \in \mathrm{I}\}$. 和上半定理证明一样,舍去若干个 G_α,便得

$$G = \sum_{\alpha \in \mathrm{I}'} \oplus G_\alpha,$$

I' 是 I 之子集. 由之又得 G_α 是单 Ω - 群.

在这里我们叙述下面定理,由于其证法与后面定理 5,6 的类似,我们略去证明.

定理 3 设 Ω - 群 G 是单 Ω - 群 $G_\alpha, \alpha \in \mathrm{I}$,之直和,其中的 *Abel* 单 Ω - 群依同构关系分成若干类,每一类中元素做成直和 $H_i, i \in N$,这样

$$G = \sum_{i \in N} \oplus H_i \oplus \sum_{\alpha \in \mathrm{I}'} \oplus G_\alpha$$

而每一 $G_\alpha, \alpha \in \mathrm{I}'$ 都是非*Abel* 单 Ω - 群,若 A 是G 之理想则有

1. $A = \sum_{i \in N} \oplus A \cap H_i \oplus \sum_{\alpha \in \mathrm{I}'} \oplus A \cap G_\alpha$;

2. 每一 $A \cap H_i, i \in N$,是与 H_i 中元素同构的*Abel* 单 Ω - 群的直和.

由之便得

定理 4 *ND* -Ω 群 G 的表为单 Ω - 群的任意两个直和必是同构的,即在此二直和之直因子间可建立一个一一对应,使相对应的直因子是同构的.

§ 3. *D* -Ω - 群

定理 5 若 G 是D -Ω - 群,则 G 是一些无真 Ω - 子群的 Ω - 群(以下简称强单 Ω - 群)$G_\alpha(\alpha \in \mathrm{I})$ 的直和,其中若 $G_\alpha, G_\beta, \alpha \neq \beta$,是非 *Abel* 的,则它们必不同构.

证明:G 是D -Ω - 群当然更是 *ND* -Ω - 群,依定理2 便有 G 是单 Ω - 群 $G_\alpha(\alpha \in \mathrm{I})$ 的直和,但 G_α 之 Ω - 子群是理想,因而 G_α 无真 Ω - 子群.

若 $G_\alpha, G_\beta, \alpha, \beta \in \mathrm{I}, \alpha \neq \beta$,是非 *Abel* 的且同构,则设 φ 是G_α 到G_β 上的一个同构对应而考虑一切形如

$$a + a\varphi, a \in G_\alpha, a\varphi \in G_\beta$$

所做成的集 H. 今证 H 是 Ω-子群. 由于 $[G_\alpha, G_\beta] = 0$, 故有, 对任意 a_i, $b \in G_\alpha$,

$$(a + a\varphi) - (b + b\varphi) = (a - b) + (a\varphi - b\varphi) = (a - b) + (a - b)\varphi$$

$$(a_1 + a_1\varphi)\cdots(a_n + a_n\varphi)\omega = (a_1\varphi)\cdots(a_n\varphi)\omega + a_1\cdots a_n\omega$$
$$= a_1\cdots a_n\omega + (a_1\cdots a_n\omega)\varphi,$$

即 H 是 Ω-子群, 因而依假设 H 是 G 之理想.

今证 $H = G_\alpha \oplus G_\beta$. 为此我们需证下面

预理 2 若 Ω-群 G 是非 *Abel* 单 Ω-群 $G_\alpha, \alpha \in \mathrm{I}$ 的直和, 则 G 的每一理想 A 必是某些 $G_\alpha, \alpha \in \mathrm{I}$, 的直和.

证明: 依假设我们有直和

$$G = \sum_{\alpha \in \mathrm{I}} \oplus G_\alpha \tag{3}$$

A 的元素关于直和 (3) 的表示式中属于 G_α 的分量全体叫做 A (关于直和 (3)) 在 G_α 中的射影. 易见理想 A 在 G_α 中的射影是 G_α 的理想, 由 G_α 之单性, 若该射影非零, 则必是整个 G_α. 今证若 A 在 G_α 之射影非零, 则 $G_\alpha \subseteq A$. 为此只需证 $A \cap G_\alpha \neq 0$.

由于 G_α 是非 *Abel* 的, 故或存在 $g_1, g_2 \in G_\alpha$ 使

$$[g_1, g_2] \neq 0, \tag{4}$$

或存在 $g_1, \cdots, g_n; h_1, \cdots, h_n \in G_\alpha, \omega \in \Omega$, 使

$$[g_1, \cdots, g_n; h_1, \cdots, h_n; \omega] \neq 0. \tag{5}$$

另一方面, 我们知道 A 在 G_α 中之射影是 G_α, 故有 $a_1 \in A$ 使 $a_1 = g_1 + g_1', g_1' \in \bar{G}_\alpha = \{G_\beta, \beta \neq \alpha, \beta \in \mathrm{I}\}$, 但

$$[a_1, g_2] = -a - g_2 + a + g_2 = -g_1' - g_1 - g_2 + g_1 + g_1' + g_2$$
$$= -g_1 - g_2 + g_1 + g_2 = [g_1, g_2]$$

而 $[a_1, g_2] \in A$, 故得 $[g_1, g_2] \in A$.

同样, 存在有 $a_i \in A, a_i = g_i + g_i', g_i' \in \bar{G}_\alpha, i = 1, 2, \cdots, n$ 依理想之定义知 A 中含有元素

$$[a_1, \cdots, a_n; h_1, \cdots, h_n; \omega] = -a_1\cdots a_n\omega - h_1\cdots h_n\omega +$$
$$(a_1 + h_1)\cdots(a_n + h_n)\omega;$$

但另一方面, 由于 $[G_\alpha, \bar{G}_\alpha] = 0$, 故

$$a_1\cdots a_n\omega = (g_1 + g_1')\cdots(g_n + g_n')\omega = g_1'\cdots g_n'\omega + g_1\cdots g_n\omega.$$

$$(a_1 + h_1)\cdots(a_n + h_n)\omega = (g_1 + h_1 + g_1')\cdots(g_n + h_n + g_n')\omega$$
$$= g_1'\cdots g_n'\omega + (g_1 + h_1)\cdots(g_n + h_n)\omega,$$

又注意 G_α 与 $\bar{G}_\alpha$ 间依元素可换(这在上面已用过了)便得

$[a_1,\cdots,a_n;h_1,\cdots,h_n;\omega] = [g_1,\cdots,g_n;h_1,\cdots,h_n;\omega] \in A$ 由(4)和(5)及刚证的便有 $A \cap G_\alpha \neq 0$,即 $G_\alpha \subseteq A$,由此即得 A 是若干个 G_α,$\alpha \in \mathrm{I}$,的直和.

现在继续证明定理5.由预理2即得 $H = G_\alpha \oplus G_\beta$,因而任取 $0 \neq b \in G_\alpha$,则 $b \in H$ 且

$$b = a + a\varphi, a \in G_\alpha$$

故有 $b = a$ 而 $a\varphi = 0$,于是 $a = 0, b = 0$,而得矛盾,故在 $G_\alpha, \alpha \in \mathrm{I}$,中不相同的非 *Abel* Ω-群必不同构.

定理6 若 G 是一些强单 Ω-群,其中非 *Abel* Ω-群中无同构者,的直和,则 G 是 D-Ω-群.

证明:设 $G = \sum\limits_{\beta \in N} \oplus H_\beta \oplus \sum\limits_{\alpha \in M} \oplus G_\alpha$,其中 $H_\beta, \beta \in N$ 是强单 *Abel* Ω-群而 $G_\alpha, \alpha \in M$,是强单非 *Abel* Ω-群,且不同的 G_α 不同构.设 A 是 G 的一个 Ω-子群.欲证定理,依定理2只需证明 Ω-子群 A 是 G 之理想.

设 $0 \neq a \in A$,则关于 G 的给定直和元素 a 有表示式

$$a = h_1 + \cdots + h_n + g_1 + \cdots + g_m \tag{6}$$

$$0 \neq h_i \in H_{\beta_i}, i \neq s \text{ 时 } \beta_i \neq \beta_s, i, s = 1,2,\cdots,n,$$

$$0 \neq g_j \in G_{\alpha_j}, j \neq t \text{ 时 } \alpha_j \neq \alpha_t, j, t = 1,2,\cdots,m.$$

称 $n + m$ 为元素 a 之长.不妨设 a 是 A 中非零元素有最小长者,考虑 a 所生成的 Ω-子群 $\{a\}$.由于 $\{a\} \subseteq A$ 而 $\{a\} \subseteq H_{\beta_1} + \cdots + H_{\alpha_n} + G_{\alpha_1} + \cdots + G_{\alpha_m}$,故 $\{a\}$ 中非零元素之长必都是 $n + m$,即任意元素 $x \in \{a\}$,x 之表示式

$$x = h_1(x) + \cdots + h_n(x) + g_1(x) + \cdots + g_n(x),$$

$h_i(x) \in H_{\beta_i}, g_j(x) \in G_{\alpha_j}$,其中等号右侧的各项或者同时为零或者同时不为零.

今证 $\{a\} \simeq H_{\beta_1}$ 利用 x 的上表示式,作

$$\varphi: x \to h_1(x), x \in \{a\}, h_1(x) \in H_{\beta_1},$$

易证这是$\{a\}$到H_{β_1}内的同态对应，依假设φ是非零的，由H_{α_1}之强单性，φ是到H_{β_1}上的对应，由上知当$x\neq 0$时$h_1(x)\neq 0$，故φ是同构对应。同时可证$\{a\}\simeq H_{\beta_i}\simeq G_{\alpha_j}, i=1,2,\cdots,n, j=1,2,\cdots,m$.

注意到H_β与G_α不会同构，而不同的G_α之间依条件也没有同构者，便得，或者在a之表示式(6)中只出现一个g_i，例如说是

$$a = g_1,$$

此时$\{a\}=G_{\alpha_1}$是G之理想；或者(6)中没有g_i出现，即

$$a = h_1+\cdots+h_n,$$

此时$\{a\}=\{h_1+\cdots+h_n\}$易见是$H=\sum_{\beta\in N}\oplus H_\beta$的理想，因为$[H,H]=0$，因而$H$之每一$\Omega$-子群都是$H$的理想。再依预理1中证过的事实便得$\{h_1+\cdots+h_n\}$是$G$之理想。

总之，我们至此得到一个结论：若A是G中任一Ω-子群而a是A的非零元素中有最小长者，则$\{a\}$是G之理想。

其次我们证明，对任意元素$b\in A$，$\{b\}$都是G的理想，设b之类似于(6)的表示式为

$$b = h_1+\cdots+h_S+g_1+\cdots+g_t,$$

$$\{b\}\subseteq H_{\beta_1}+\cdots+H_{\beta_S}+G_{\alpha_1}+\cdots+G_{\alpha_t},$$

设b_1是$\{b\}$中有最小长的非零元素，则依上b_1之类似于(6)之表示式可能有两种情形

(一)或者$b_1=g_1$，则此时理想$\{b_1\}$是G之直因子，故有$\{b\}=\{b_1\}\oplus B_1, B_1\subseteq H_{\beta_1}+\cdots+H_{\beta_S}+G_{\alpha_2}+\cdots+G_{\alpha_t}$;

(二)或者$b_1=h_1(b_1)+h_2(b_1)+\cdots+h_n(b_1), h_i(b_1)\neq 0, i=1,2,\cdots n$.由上知$\{b_1\}$是理想，依定理2知它是$G$之直因子，今证

$$G=\{b_1\}\oplus\sum_{\beta\neq\beta_1,\beta\in N}H_\beta\oplus\sum_{\alpha\in M}\oplus G_\alpha \tag{7}$$

约定若用K表示出现在上等式右侧的某一理想时，则用$\overline{K}$表示该右侧中除去K外其余理想的和，首先G是(7)的右侧中所有理想之和，其次，注意到原给G的直和分解以及$[H,H]=0$，易见对(7)之右侧中任意K都有$[K,\overline{K}]=0$.设

$$0 = x+h_2+\cdots+h_n+\cdots+h_\beta+g_\alpha+\cdots(\text{有限和})$$

$$x\in\{b_1\}, h_i\in H_{\beta_i}, i=2,\cdots n, h_\beta\in H_\beta,\cdots,g_\alpha\in G_\alpha,\cdots$$

必有此等式中每一项都是零, 这是因为

$$\begin{aligned}0 &= h_1(x)+\cdots+h_n(x)+h_2+\cdots+h_n+h_\beta+\cdots+g_\alpha+\cdots \\ &= h_1(x)+(h_2(x)+h_2)+\cdots+(h_n(x)+h_n)+h_\beta+\cdots+g_\alpha+\cdots\end{aligned} \tag{8}$$

因而 $h_1(x)=0$, 但 b_1 是 Ω - 子群 $\{b\}$ 中有最小长之非零元素而 $x\in\{b_1\}$, 所以依上有 $h_2(x)=\cdots=h_n(x)=0$, 再由等式 (8) 便有 $h_2=\cdots=h_n=h_\beta=\cdots=g_\alpha=\cdots=0$, 即证得 G 之每一元素在(7) 之右侧的和中有唯一表示式总起来便有直和(7), 此时有

$$\{b\}=\{b_1\}\oplus B_1,$$

$$\begin{aligned}B_1 &= \{b\}\cap\Big(\sum_{\beta\neq\beta_1,\beta\in N}\oplus H_\beta\oplus\sum_{\alpha\in M}\oplus G_\alpha\Big) \\ &\subseteq H_{\beta_2}+\cdots+H_{\beta_S}+G_{\alpha_1}+\cdots+G_{\alpha_t}\end{aligned}$$

总之, 无论这两种中那一种情形都有

$$\{b\}=\{b_1\}\oplus B_1:$$

$\{b_1\}$ 是 G 的理想而 B_1 中每一元素之长都较 b 的小, 依归纳法假设 B_1 该是 G 之理想, 故得 $\{b\}$ 是 G 之理想.

最后再用一下定理 2 便知 G 是 D - Ω - 群, 定理全部证完.

§ 4. *SD* - Ω - 群, *R* - Ω - 群

定理7 Ω - 群 G 是 SD - Ω - 群当且仅当 G 是一些强单 Ω - 群的直和.

证明: 若 G 是 SD - Ω - 群, 则依定理 1, G 是 ND - Ω - 群, 因而是单 Ω - 群 G_α, $\alpha\in\mathrm{I}$, 的直和. 若 A 是某一 G_α 的真 Ω - 子群, 则依 G 之 SD 性, 必有 G 的一个理想 B, 使

$$G=A+B \quad 及 \quad A\cap B=0,$$

因而

$$G_\alpha=A+G_\alpha\cap B \quad 及 \quad A\cap(G_\alpha\cap B)=0,$$

故知 $G_\alpha\cap B$ 是 G_α 之真理想, 但这与 G_α 之单性矛盾, 故每一 G_α 都是强单 Ω - 群.

反之, 若 G 是强单 Ω - 群 G_α, $\alpha\in\mathrm{I}$, 的直和而 A 是 G 的任一 Ω - 子群, 则有

$$G=A+\sum_{\alpha\in\mathrm{I}}\oplus G_\alpha.$$

将 I 良序之，若对所有 $\alpha < \mu$（一固定足码）G_α 之取舍已确定，而设 M 是保留下来之 G_α 的足码集，则当

$$A + \sum_{\alpha \in M} G_\alpha \text{ 与 } G_\mu \text{ 之交} = 0$$

时保留 G_μ，否则舍去 G_α，此时依 G_μ 之强单性必有

$$G_\mu \subseteq A + \sum_{\alpha \in M} G_\alpha,$$

这样设所有保留下来的 G_α 之足码集是 I′，则易见（和定理 2 类似）

$$G = A + \sum_{\alpha \in I'} G_\alpha \quad \text{而 } A \cap \sum_{\alpha \in I'} G_\alpha = 0,$$

显然 $\sum_{\alpha \in I'} G_\alpha$ 是理想.

定理 8　Ω - 群 G 是 R -Ω - 群当且仅当 G 是一些强单 Ω - 群 G_α，$\alpha \in$ I，其中无同构的非 *Abel* 强单 Ω - 群的直和.

证明：若 G 是 R -Ω - 群，则依定理 1，G 是 SD -Ω - 群，因而依定理 7，G 是强单 Ω - 群 G_α，$\alpha \in$ I，的直和. 若非 *Abel* Ω - 群 G_α，G_β，$\alpha \neq \beta$，α，$\beta \in$ I 同构，则必将引出矛盾. 为此，设 φ 是 G_α 到 G_β 上的一个同构对应，则一切形如 $a + a\varphi$，$a \in G_\alpha$，的元素集 H，如定理 5 中所证，是 Ω - 子群. 今考虑Ω - 子群

$$A = H + \sum_{\gamma \neq \alpha,\, \gamma \neq \beta,\, \gamma \in I} G_r$$

如定理 5 中所证

$$G_\alpha \cap H = 0,$$

故从而知 A 是 G 之真 Ω - 子群，但 A 所生成的理想，依预理 2，等于 G，所以 A 不能是 G 之正则因子，故 G_α，$\alpha \in$ I，中无同构的非 *Abel* Ω - 群.

反之，若 G 是定理中所述的直和，则 G 是 D -Ω - 群当然更是 R - Ω - 群.

§ 5. F -Ω - 群，F -ΩA - 群

用 ΩA - 群表示具有 *Abel* 加群的 Ω - 群.

一个 Ω - 群（ΩA - 群）G 称为其 Ω - 子群（ΩA - 子群）B_1，B_2 的自由和，并记成

$$G = B_1 *_\Omega B_2 (G = B_1 *_{\Omega A} B_2) \tag{9}$$

如果 $G = \{B_1, B_2\}$，并且若任意给定 $B_i (i = 1, 2)$ 到任意给定的 Ω - 群

(ΩA-群)G' 中的同态对应 $\varphi_i(i=1,2)$ 则必存在,且只有一个,Ω-群(ΩA-群)G 到 Ω-群(ΩA-群)G' 中的同态对应 φ,它在 $B_i(i=1,2)$ 上与 $\varphi_i(i=1,2)$ 重合.

若已知有(9),取 $B=B_1$ 而命 φ_1 是 B_1 到 B 上的恒等自同构而 φ_2 是 B_2 到 B 内的零同态,依自由和之定义必有 G 到 B 上的一个同态 φ. 由于 φ 在 B_1 上与 φ_1 重合,故 $B_1\cap C=0$,这里的 C 是 φ 之核心. 依[1]中结果 C 是 G 之理想. 另一方面,φ 在 B_2 上与 φ_2 重合,故 $B_2\subseteq C$. 由于 B_1,B_2 所处地位的对称性,这就证明了下面

预理 3 F-Ω-群必是 R-Ω-群;F-ΩA-群必是 R-ΩA-群.

不难看到(或参看[6]),若 Ω-群 G 有(9),且 $B_1\neq 0,B_2\neq 0$ 则必 $[B_1,B_2]\neq 0$,若 ΩA-群 G 有(9)且 Ω 非空时亦必有 $[B_1,B_2]\neq 0$. 今证

定理 9 G 是 F-Ω-群当且仅当 G 是一个强单 Ω-群;当 Ω 非空时,G 是 F-ΩA-群当且仅当 G 是一个强单 ΩA-群.

证明:设 G 是 F-Ω-群,不妨认为 $G\neq 0$,依预理 3,G 是 R-Ω-群,因而有

$$G=\sum_{\alpha\in I}\oplus G_\alpha \tag{10}$$

其中 G_α 是强单 Ω-群,$G_\alpha\neq 0,\alpha\in I$. 若 I 含有两个以上元素,今任取其中一个 G_α,由于 G 有性质 F,故必有一 Ω-子群 B,显然 $B\neq 0$,使

$$G=G_\alpha *_\Omega B \quad 因而[G_\alpha,B]\neq 0,$$

若 G_α 是 $Abel$-Ω-群,则依(10)有 $[G_\alpha,G]=0$,与上矛盾. 若 G_α 是非 $Abel$-Ω-群,则由 $[G_\alpha,B]\neq 0$,知 B 关于直和(10)在 G_α 中的射影必不为零,因而依预理 2,B 在 G 中所生成的理想包含 G_α,但这与预理 3 之证明相矛盾,故 G 是一个强单 Ω-群.

反之,一个强单 Ω-群显然是 F-Ω-群.

完全类似的讨论便得到定理中的第二个结论.

§6. 一些推论

由于强单群显然是且仅是元数为素数的循环群,所以没有非 $Abel$ 的强单群,这样对于群来说由上面定理 5,6,7,8 便得 $[SD]=[R]=[D]$ 且 D-群是且仅是一些元数为素数的循环群的直和,这就是[2]、[5]中的结

果，而从形式来看这里的结果较[5]中关于D-群的刻画更好一些．显然，[3]中关于ND-带算子的群的结果是定理2的推论，而[2]中关于F-群之结论是定理9的推论．

由于强单幂结合环显然是且仅是加群为素元数的循环群的零环以及素域，并且把环看成Ω-群时易见零环是*Abel*的而素域是非*Abel*的，这样由定理5和6便得[4]中关于D-环的结果．

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О ПРЯМЫХ СЛАГАЕМЫХ В МУЛЬТИОПЕРАТОРНЫХ ГРУППАХ

Резюме Группы с мультиоператорами введены в работе[1]. Именно, группа G, аддитивно записанная, хотя не обязательно коммутативная, называется группой с системой мультиоператоровΩили, короче, Ω-группой, еспи всякий оператор $\omega \in \Omega$ является n-арной алгебраической операцией, заданной в G, $n \geqslant 1$, причем выполняется требование

$$0\,0\cdots 0\omega = 0$$

Обозначим ниже указанные свойства, которыми могут обладать данныеΩ-группы, через следуюшие буквы:

ND : каждый идеал является прямым слагаемым;

SD : каждая Ω-подгруппа является полупрямым слагаемым;

R : каждая Ω-подгруппа является правильным слагаемым в смысле Головина;

D : каждая Ω-подгруппа является прямым слагаемым;

F : каждая Ω-подгруппа является свободным слагаемым.

Полные определения этих свойств и встречающихся ниже понятий найдутся в работах[1]、[8]、[7], Ω-группа, обладающая свойством M обозначается через M-Ω-группу,

Доказаны нами следующие теаремы:

Теарема 1 G естьND-Ω-группа тогда и только тогда, когда G есть

прямая сумма простых Ω-групп Притом любые два таких прямых разложения Ω-группы G изоморфны между собой

Ω-группа без нетривиальных Ω-подгрупп называется сильно простой,

Теарема 2 G есть SD-Ω-группа тогда и только тогда, когда G есть прямая сумма спльных простыхΩ-групп,

Теарема 3 Дпя Ω-групп свойства R и D эквивалентны между собой.

Теарема 4 G есть D-Ω-группа тогда и только тогда, когда G есть прямая сумма сильно простых абелевых Ω-групп и сильно простых неизоморфных друг другу не-абелевых Ω-групп.

Ω-группа с абелевой аддивной группой будет называться ΩA-группой.

Теарема 5 G есть F-Ω-группа(F-ΩA-группа)тогда и только тогда, когда G есть сильно простая Ω-группа(сильно простая ΩA-группа).

Так как к числу Ω-групп принадлежат группы, кольца, а также группы и кольца с операторами, то из наших теарем, как простые следствия, вытекают все известные до сих пор аналогичные результаты для групп и колец, с операт-орами или без операторов(см.[3][4][5][2]), а также ранее неизвестные результаты для них.

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几类非结合环的局部幂零性和 Levitzki 根*

On Local Nilpotency of Some Classes of Non-Associative Rings and their Levitzki Radicals

在[3]中对于交错代数,Jordan 代数以及代数的 Lie 代数证明了局部有限代数借助局部有限代数所得的扩张仍是局部有限的.由之可得上述三类代数的有关局部幂零性的一些结果.本文的目的在于把上述结果推广到交错环,Jordan 环及代数的 Lie 环上去.从而可得这三类环的有关局部幂零性的一些结果.

设环 A 以 I 为其算子环而 I 是可换主理想环.说 A 是模有限环,如果把 A 看作 I - 模时,它有有限生成元系.当 I 是域时这就是代数的有限维的概念.对一般环可取 I 为整数环,此时模有限环是指其加群为有限生成的 Abel 群的环.说环 A 是局部模有限的,如果 A 的任意有限个元素生成的子环是模有限的.现在来证下面三个预理,其中环 A 都以可换主理想环 I 为算子环.

预理 1 若交错环 A 之理想 R 及商环 A/R 都是局部模有限的,则 A 也是.

预理 2 若 Jordan 环 A 的理想 R 及商环 A/R 都是局部模有限的,且 A 之特征 $\neq 2$(即 A 的加群中没有阶为 2 的元素),则 A 的任意有限个元素生成的子环 C 中必有一有限生成的 I - 子模 D,对任意 $x \in C$ 有 $2^n x \in D$,其中 $n = n(x)$ 是自然数.

* 1964 年 2 月 20 日收到.1979 年 5 月 18 日改为此简报形式.

说 Lie 环 A 为代数的，若对任意 $x, y \in A$，存在一自然数 $k = k(x, y)$，$x, xy, \cdots, xy^k = (xy^{k-1})y$ 中最后一个元素可表为前面 k 个元素的线性和，系数在 I 中.

预理 3　若代数的 Lie 环 A 的理想 R 和商环 A/R 都是局部模有限的，则 A 也是.

这三个预理的证明顺序与[3]中 §2 预理 2,3,4 的证明类似. 为了说明问题在这里给出预理 3 的完整证明，其他的证明较长请参看[3].

预理 3 的证明：显然只需讨论 $\bar{A} = A/R$ 是模有限的情形. 设 $\bar{a}_i, i = 1, 2, \cdots, n$ 是有限生成 I-模 $\bar{A}$ 的一组生成元，则有

$$a_i a_j = \sum_{k=1}^{n} f_{ijk} a_k + x_{ij}, i, j = 1, 2, \cdots, n \tag{1}$$

其中 a_i 是剩余类 $\bar{a}_i$ 的某一取定的代表，$f_{ijk} \in I, x_{ij} \in R$. 为了证明 A 是局部模有限的，只需证 $a_i, i = 1, 2, \cdots, n$ 和 R 中一包含 $x_{ij}, i, j = 1, 2, \cdots, n$ 的模有限子环 X 在一起于 A 中生成模有限子环，这是因为我们有可换主理想环上有限生成的模的任一子模也是有限生成的这一熟知事实.

令 $x_j, j = 1, 2, \cdots, m$ 是模 X 的一组生成元. 令 M 是由 $a_i, i = 1, \cdots, n$ 及 $x_j, j = 1, \cdots, m$ 组成的集合. 用 M 中的元素可以作任意长度的非结合字. 字 b 之长用 $d(b)$ 来表示，下面将用

$$b = b_1 + b_2 + \cdots + b_i + \backsim$$

表示：在环 A 中字 b 是字 b_i 及一些其长度小于 $d(b), d(b_i)$ 的字的和，因之等式 $b = \backsim$ 也是有意义的. 令 W 是所有至少含一个 x_j 的字的全体. 令 W_k 表 W 中所有长度 $\leqslant k$ 的字的全体. W_k 是 R 中的有限子集，故生成 R 的模有限子环 B_k. 若 N 是所有 a_i 生成的 I-模，则显然 $B_k + N$ 是 A 的有限生成子模，且包含所有的 a_i 和 x_j. 因而要证所有 a_i, x_j 在 A 中生成模有限子环，只需证存在一自然数 k，使 $B_k + N$ 是子环. 而为此利用(1)只需证明 $W \subseteq B_k$.

因为 Lie 环 A 是代数的，且 X 是有限生成模，故对每一 a_i，存在自然数 n_i，有

$$x a_i^{n_i} = \backsim \quad \forall x \in X \tag{2}$$

和通常一样，其中 $xa^n = (xa^{n-1})a$. 取 $k = \sum_{i=1}^{n} n_i + 1$. 对字之长度作归纳

法来证 $W \subseteq B_k$.

设 $b \in W$. 易见我们有

$$(ba_j)a_i = -(a_ja_i)b + (ba_i)a_j = (ba_i)a_j + \backsim \qquad (3)$$

这里我们利用了(1). 若 $d(b) \leqslant k$, 则 $b \in W_k \subseteq B_k$. 设 $d(b) > k$.

若在字 b 中只有一个形如 x_i 的因子, 说是 x_1, 由于 Lie 环是反交换的, 故可设 $b = x_1a_{i_1}\cdots a_{i_p}$(这是右正规乘积, 即 $x_1a_{i_1}\cdots a_{i_{p-1}}a_{i_p} = (x_1a_{i_1}\cdots a_{i_{p-1}})a_{i_p}$), 否则易见将有 $b = \backsim$. 因为 $d(b) > k = \sum n_i + 1$, 故某一 a_i, 说是 a_1 在 b 中至少出现 n_1 次. 利用变换(3) 将因子 a_1 不断地向前移动, 最后可得

$$b = xa_1^{n_1}a_{i_1}\cdots a_{i_t} + \backsim$$

由之据(2) 便得 $b = \backsim$, 依归纳假设即知 $b \in B_k$.

假若在 b 中出现两个以上形如 x_i 的因子, 则可设 $b = b_1b_2a_{i_1}\cdots a_{i_p}$, 其中 $b_1, b_2 \in W$. 用等式

$$(b_1b_2)a = (ab_2)b_1 + (b_1a)b_2$$

去变换便得 $b = \sum b_{i_1}b_{i_2}$, 其中 $b_{i_1}, b_{i_2} \in W$ 且其长度显然小于 $d(b)$. 依归纳假设即知这些 b_{i_j} 属于 B_k, 因而 $b \in B_k$. 这样就证得 $W \subseteq B_k$.

利用这些结果, 转来证明下面三个预理:

预理 4 设 A 是交错环. 若 A 的理想 R 及商环 A/R 都是局部幂零的, 则 A 也是.

证明: 由于局部幂零环必是局部模有限环, 故依预理 1 知 A 是局部模有限的. 另一方面, 易见 A 是幂零元素的. 故 A 中任意有限个元素所生成的子环 C 是幂零元素的且是模有限的. 但主理想环 I 上的有限生成模对子模有极大条件, 因而环 C 对子环有极大条件. 依[4] 中结果: 对幂零子环有极大条件的幂零元素交错环是幂零的, 知 C 是幂零的.

预理 5 设 A 是 Jordan 环且 A 之特征 $\neq 2$, 若 A 的理想 R 及商环 A/R 都是局部幂零的, 则 A 也是.

证明: 设 C 是 A 中有限个元素生成的子环. 易见 C 是幂零元素的. 下面证明 C 中有极大的幂零子环. 依 Zorn 引理, 为此只需证明 C 中任一幂零子环链 $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_m \subseteq \cdots$ 之并 B 也是幂零子环. 依预理 2 在 C 中存在一有限生成的子模 D, 对任意 $x \in C$ 有 $2^nx \in D$. 其中 $n = n(x)$

是自然数.作模 D 的子模链 $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_m \subseteq \cdots$ 其中 $D_i = B_i \cap D$.由于模 D 对子模有极大条件,故 $\exists\, m$ 使 $B_m \cap D = B_s \cap D, s \geqslant m$.这说明对任意 $x \in B_s, s \geqslant m$,必有 $-n = n(x)$,使得 $2^n x \in B_m$.由此注意到 A 的特征 $\neq 2$ 便得,若 B_m 之幂零指数为 k,则对任意 $s \geqslant m, B_s$ 的幂零指数也是 k,即得其并 B 是幂零的.

另一方面,依[4]中结果,幂零元素 Jordan 环不能有异于 C 的极大幂零子环.故得 C 本身是幂零的.

预理 6 设 A 是 Lie 环且满足 Engel 条件(即对任意 $x, y \in A$ 有自然数 $n = n(x, y)$,使 $xy^n = 0$).若 A 的理想 R 及商环 A/R 是局部幂零的,则 A 也是.

证明:有 Engel 条件的 Lie 环当然是代数的,故依预理 3, A 是局部模有限的.设 C 是 A 中有限个元素生成的子环.由上知模 C 是有限生成的.设 $c_1, \cdots, c_t$ 是 C 的生成元,则由 Engel 条件,对任意 $x \in C$,有 n_i 使 $c_i x^{n_i} = 0$.取 $n = \max\limits_i n_i$,则有 $Cx^n = 0$.即 C 是幂零元素 Lie 环.另一方面知 C 对子环有极大条件,依[5]中结果知 C 是幂零的.

由这几个预理依通常的方法便得下面两个定理.

定理 1① 设 A 是交错环,或特征不为 2 的 Jordan 环,或有 Engel 条件的 Lie 环,则 A 中存在一最大的局部幂零理想 R(称之为 A 的 Levitzki 根),它包含 A 中一切局部幂零理想,且商环 A/R 的 Levitzki 根为零(称之为 Levitzki 半单的).

定理 2 设 A 是交错环,或特征不为 2 的 Jordan 环,或有 Engel 条件的 Lie 环.若 A 中有一递增可解链(即有一子环链 $0 = A_0 \subset A_1 \subset \cdots \subset A_\alpha \subset \cdots \subset A_\tau = A$,其中当 α 是极限数时 $A_\alpha = \bigcup\limits_{\beta<\alpha} A_\beta$,而当 $\alpha - 1$ 存在时有 $A_\alpha^2 \subseteq A_{\alpha-1}$),则 A 是局部幂零的.

定理 1① 中关于交错环部分是 Levitzki [1] 中关于结合环的结果的推广,而关于 Lie 环的结果是[6]中定理 3,4 的推广.

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① 改写时附记.定理 1 中关于 Jordan 环部分以及预理 2 也是以下两文中的主要结果:Жевлаков К.А., Разрешимость и нильпотентность йордановых колец, *Алгебра и логика*, 1966, 5(3):37 ~ 58; Tsai C.E., The Levitzki radical in Jordan rings, *Proc. Amer. Math. Soc.*, 1970, 24: 119 ~ 123.

定理 2 是与[2] 中关于群的定理相平行的结果. 其中关于 Lie 环的结果与[7] 的主要结果相类似, 后者断言: 设 A 是有 n 次 Engel 条件的 Lie 环且其特征与 $n!$ 互质. 若 A 还是可解的, 则 A 是幂零的.

下面给一个 Lie 环的例子说明定理 1,2 中有关 Lie 环的 Engel 条件是不能去掉的.

令 $e_i, i = 0,1,2,\cdots$ 是整数环上自由模 A 的基. 规定 $e_i^2 = 0, i = 0, 1, 2, \cdots; e_ie_j = -e_je_i, i, j = 0, 1, 2, \cdots; e_ie_j = 0, i, j = 1, 2, \cdots; e_0e_i = e_{i+1}, i = 1, 2, \cdots$, 则 A 成为 Lie 环. 设 R 是 $e_i, i = 1, 2\cdots$ 生成的子模, 则易见 R 是 A 的最大局部幂零理想, 但 A/R 仍是幂零的.

最后我们指出, 由预理 4,5,6 可直接得到下述结果.

定理 3① 有理想化子条件的交错环, Jordan 环或 Lie 环必是局部幂零的.

此定理在[8] 中是用另外方法证明的.

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① 改写时附记. 在 Hartley B., Locally nilpotent ideals of a Lie algebra, *Proc. Cambridge Phil. Soc.*, 1967, 63:257 ~ 272 中叙述的一个结果: 有理想化子条件的, 特征为零的域 F 上的 Lie 代数是局部幂零的, 是这里定理 3 的一个特例.

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每一子代数都是理想的代数

On Algebras in Which Every Subalgebra is an Ideal

Those groups in which every subgroup is normal and those loops in which every subloop is normal are determined in [1] and [2] respectively. The problem of the determination of those rings in which every subring is an ideal (abbreviated as H-ring) is more complicated and unsolved. There are several results dealing with H-rings of special kinds. Those H-rings with a single generator were given in [3, 7]. Those rings in which every subgroup is an ideal were described in [4]. In [5] and several related papers, F. Szász discussed a family of rings that may be regarded as special kind of H-rings.

The purpose of this paper is to determine those H-rings with special operator domain that we will call H-algebra. The isomorphism problem of H-algebras is also discussed. Our main theorem is the following.

Theorem 1 Let A be an alternative ring or a Jordan algebra over the field Φ(in the case of Jordan algebra, we assume that the characteristic of $\Phi \neq 2$). ①Every subalgebra of A is an ideal if and only if A is one of the following algebras:

1) one-dimensional idempotent algebra;

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① For basic definitions and theorems about alternating rings and Jordan algebras, we refer to A. G. Kuroš, Lectures on general algebra, Fizmatgiz, Moscow, 1962(Russian), and A. A. Albert, A structure theory for Jordan algebras, Ann. of Math. 1947, 48(2): 546 ~ 567

2) null algebras i.e. algebras with null multiplication;

3) an algebra with basis $a_0, a_i, i \in I$ (where I is a nonempty index set of arbitrary cardinal number) and with the following multiplication table:

$$a_0^2 = a_0 a_i = a_i a_0 = 0, a_i a_j = \alpha_{ij} a_0,$$

$$\alpha_{ij} \in \Phi, i, j \in I.$$

Moreover, if N is a finite subset of I, then $\sum_{i,j \in N} \alpha_{ij} x_i x_j$ is nondegenerate in the sense that $\sum_{i,j \in N} \alpha_{ij} \alpha_i \alpha_j = 0$, $\alpha_i \in \Phi$, $i \in N$ implies $\alpha_i = 0$, $i \in N$.

4) direct sums of algebras of 1), 2), 3) types with at most one from each.

Before proving Theorem 1, we first give two simple corollaries.

Corollary 1 H-algebras are associative, and if A does not contain idempotent elements, then $A^3 = 0$.

Corollary 2 If every second degree polynomial over Φ is reducible, then every H-algebra A splits as a direct sum

$$A = A_1 \oplus A_2 \oplus A_3,$$

where A_1 is a one-dimensional idempotent algebra, A_2 is a null algebra and A_3 is a two-dimensional algebra generated by a nilpotent element of nilpotency three; or possibly with some of the three summands neglected. Hence, every H-algebra over such a field is commutative.

Since the only nondegenerate quadratic form over Φ is of rank one, we see that the I in 3) of Theorem 1 contains at most one index and hence Corollary 2 follows.

In case the field Φ does not have the property of Corollary 2, it is easy to construct noncommutative H-algebras by Theorem 1.

§ 1. The proof of Theorem 1

In this section, we let A be either an alternative ring or a Jordan algebra over a given field Φ. It is well known that A is power-associative, i.e. the subalgebra $\{a\}$ generated by an element $a \in A$ is always associative.

We first prove several lemmas.

Lemma 1 Let A be an H-algebra; then A is locally finite.

Proof Let a be any element in A. If $\{a\}$ is infinite-dimensional then $a, a^2, \cdots, a^n, \cdots$ are linearly independent. But, by the definition of H-algebra, $\{a^2\}$ is an ideal, hence $a^3 = a \cdot a^2 \{a^2\}$, which is a contradiction. Hence $\{a\}$ is finite-dimensional. Now let a_i, $i = 1, 2, \cdots, m$ be finite elements in A, we see easily that

$$\{a_1, a_2, \cdots, a_m\} = \{a_1\} + \{a_2\} + \cdots + \{a_m\}$$

is again finite-dimensional, i.e., A is a locally finite. This completes the proof.

Lemma 2 If A is an H-algebra with at least one element that is not nilpotent, then $A = \{e\} \oplus A_1$ with $e^2 = e$ and A_1 is nilpotent.

Proof Let $a \in A$ be an element in A that is not nilpotent. By Lemma 1, $\{a\}$ is finite-dimensional and hence $\{a\}$ contains at least one idempotent element e. We will show that $\{e\}$ is a direct summand in A.

Let $A_1 = \{x \in A \mid ex = 0\}$, we claim that $A = \{e\} \oplus A_1$. Since $\{e\}$ is an ideal and e is the unit element in $\{e\}$, for any $x, y \in A_1$, we have

i) in case A is an alternative ring:

$$xe = e(xe) = (ex)e = 0,$$
$$e(xy) = (ex)y - x(ey) + (xe)y = 0,$$

ii) in case A is a Jordan algebra; in the identity

$$[(yz)x]w + [(zw)x]y + [(wy)x]z$$
$$= (yz)(xw) + (zw)(xy) + (wy)(xz), \tag{1}$$

put $w = z = e$, and we have $e(xy) = 0$.

In every case $xy \in A_1$, namely, A_1 is a subalgebra. Since $\{e\} \cap A_1 = 0$ we see that

$$\{e, A_1\} = \{e\} \oplus A_1.$$

On the other hand, to any $a \in A$, we have

$$a = (a - ea) + ea, e(a - ea) = ea - e(ea) = ea - ea = 0,$$

i.e. $A = \{e, A_1\} = \{e\} \oplus A_1$.

It is obvious that A_1 is also an H-algebra. If A_1 contains elements that are not nilpotent, by the above argument we have $A_1 = \{e_1\} \oplus A_2, e_1^2 =$

e_1. That is

$$A = \{e\} \oplus \{e_1\} \oplus A_2.$$

Let $f = e_1 + e_2$; then f is idempotent and $\{f\}$ is one-dimensional. On the other hand $e = ef \in \{f\}$ and $e_1 = e_1 f \in \{f\}$ and e, e_1 are linearly independent, which is a contradiction. Hence A_1 must be nilpotent. The proof is complete.

Lemma 3 Let A be an H-algebra, then every nilpotent element $a \in A$ is of nilpotency $\leqslant 3$, i.e. $a^3 = 0$.

Proof Suppose there is an element $a \in A$, $a^n = 0$, $a^{n-1} \neq 0$ and $n \geqslant 4$. Then $(a^m)^2 = 0$ with $m = [(n+1)/2]$, $m + 1 \leqslant n - 1$. But $a^{m+1} = a^m \cdot a \in \{a^m\}$ and $\{a^m\}$ is one-dimensional, so that we have $a^{m+1} = \alpha a^m$, $\alpha \in \Phi$. If $\alpha \neq 0$ then a is not nilpotent; if $\alpha = 0$ then $a^{n-1} = 0$, which is a contradiction. Hence $n \leqslant 3$, and the proof is complete.

Lemma 4 Let a be an element in an H-algebra A with the property $a^2 = 0$, then $aA = Aa = 0$.

Proof Let a be any element in A, we will show that $a \cdot b = ba = 0$. By Lemma 2, we may assume b is nilpotent and hence, by Lemma 3, $b^3 = 0$. Since $\{a\}$ is one-dimensional, $ab = \alpha a$, $\alpha \in \Phi$. We have the following two cases:

i) Alternative ring case: by the Artin theorem (see p. 229, A. G. Kuroš's Lectures on general algebra) we know that every subalgebra generated by two elements is associative; hence

$$0 = ab^3 = \alpha^3 a \text{ implies } \alpha = 0.$$

ii) Jordan algebra case: put $w = a$, $x = y = z = b$ in the identity (1) we have

$$2[(ab)b]b = -ab^3 + 3(ab)b^2.$$

But $b^3 = 0$, and $ab = \alpha a$, we have $3(ab)b^2 = 2\alpha^3 a$ and hence

$$9[((ab)b^2)b]b^2 = 6\alpha^3(ab)b^2 = 4\alpha^6 a.$$

On the other hand, put $x = b$, $y = ab$, $z = w = b^2$, we have

$$2[((ab)b^2)b]b^2 = -(ab)b^5 + 2[(ab)b^2]b^3 + [(ab)b]b^4 = 0.$$

Hence $4\alpha^6 = 0$, $\alpha = 0$ for the characteristic of $\Phi \neq 2$; thus we show in both

cases $ab = 0$. Similarly, we may also show that $b \cdot a = 0$. The proof is complete.

Lemma 5 Let A be an H-algebra and $a, b \in A$ such that $a^2 \neq 0, a^3 = 0$ and $b^2 \neq 0, b^3 = 0$. Then

i) a, a^2 are linearly independent;

ii) a, a^2, b, b^2 are linearly dependent;

iii) $b^2 = \alpha a^2, ab = \beta a^2, ba = \gamma a^2, \alpha, \beta, \gamma \in \Phi$.

Proof i) is obvious. We prove ii).

Suppose a, a^2, b, b^2 are linearly independent, then $\{a, b\} = \{a\} \oplus \{b\}$, because $\{a\}$ and $\{b\}$ are ideals with basis a, a^2 and b, b^2 respectively, Now, $\{a + b\}$ is a two-dimensional subalgebra with $(a + b)$ and $a^2 + b^2$ as basis. Since $\{a + b\}$ is an ideal, we have $a^2 = a(a + b) \in \{a + b\}$ and hence in the two-dimensional subalgebra $\{a + b\}$ we get three linearly independent elements $a + b, a^2, b^2$, which is a contradiction.

The proof of iii). If a, a^2, b are linearly dependent, then $b = \alpha_1 a + \alpha_2 a^2$. By Lemma 4, we have $b^2 = \alpha_1^2 a^2$. If a, a^2, b are linearly independent, we have

$$b^2 = \alpha_1 a + \alpha_2 a^2 + \beta_1 b.$$

Now, if $ab = 0$, then, multiplying by b on both sides, we get $\beta_1 b^2 = 0$, namely, $\beta_1 = 0$; similarly, we get $\alpha_1 a^2 = 0$ and hence $\alpha_1 = 0$ and $b^2 = \alpha_2 a^2$. If $a \cdot b \neq 0$, we get

$$\alpha_1 ab = -\beta_1 b^2, \beta_1 ab = -\alpha_1 a^2.$$

It is obvious that either α_1, β_1 are both nonzero or they are both zero. If $\alpha_1 = \beta_1 = 0$, we again have $b^2 = \alpha_2 a^2$; if $\alpha_1 \neq 0, \beta_1 \neq 0$ we have

$$b^2 = \left(\frac{\alpha_1}{\beta_1}\right)^2 a^2.$$

We have already shown the first part of iii), for the rest, they follow from Lemma 4 and the fact that $ab \in \{a\}$ and is a linear combination of a and a^2.

The proof of Theorem 1. Let A be an H-algebra; we have the following cases:

1) If A contains at least one element that is not nilpotent, then, by Lemma 2, $A = \{e\} \oplus A_1$, $e^2 = e$ and A_1 is a nilpotent H-algebra.

2) If all elements in A are nilpotent, and A is not a null algebra, then, by Lemmas 3, 4, 5, A is nilpotent and $A^3 = 0$ with at least one element a, $a^2 \neq 0$.

3) Let A be an H-algebra such that $A^3 = 0$ and there is an $a \in A$ with $a^2 \neq 0$. Then, by Zorn's lemma, there is a maximal null subalgebra A_2'. Let $a_0 = a^2$, we see easily that $a_0 \in A_2'$ by Lemma 4. Let $a_0, b_t, t \in T$ (T is an index set) consist of a basis for A_2'; and $a_0, b_t, t \in T; a_i, i \in I$ (I is another index set) form a basis of A. We may assume $a = a_1, 1 \in I$, i.e. $a_1^2 = a_0$. It follows from the choice of A_2' and Lemma 4 that any nonzero element in the linear subspace generated by $a_i, i \in I$ is of nilpotency $= 3$. Furthermore, by Lemma 5, the linear subspace generated by $a_0, a_i, i \in I$ is a subalgebra A_3. Let A_2 be the null subalgebra generated by $b_t, t \in T$, then we have trivially

$$A = A_2 \oplus A_3.$$

Moreover, by Lemmas 4 and 5, A_3 has the following multiplication table with respect to the basis $a_0, a_i, i \in I$,

$$a_0^2 = a_0 a_i = a_i a_0 = 0, a_i a_j = \alpha_{ij} a_0,$$

$$\alpha_{ij} \in \Phi, i, j \in I.$$

Now, we claim that the quadratic form $\sum_{i,j \in N} \alpha_{ij} x_i x_j$ given by the structural constant α_{ij} is nondegenerate for any finite subset $N \subseteq I$. Let $\alpha_i \in \Phi, i \in N$ be not all zero, then $c = \sum_{i \in N} \alpha_i a_i \neq 0$ and hence $c \neq 0$. But we have

$$c^2 = \Big(\sum_{i,j \in N} \alpha_{ij} \alpha_i \alpha_j \Big) a_0.$$

That is,

$$\sum_{i,j \in N} \alpha_{ij} \alpha_i \alpha_j \neq 0$$

for $\alpha_i, i \in N$, not all zero. The necessary part of the theorem is now completely proved. Now, we shall show that it is also sufficient.

First we observe that algebras of 1), 2) types are H-algebras and they

are all associative. We claim that all algebras of type 3) are also H-algebras. To show that, we need only to show that $\{c\}$ is an ideal for every $c \in A_3$. In terms of the given basis,

$$c = \alpha_0 a_0 + \sum_{i \in N} \alpha_i a_i, 0 \neq \alpha_i \in \Phi, i \in N, \tag{2}$$

where N is either the empty set or a finite subset of I. If N is empty, we have $\{c\} = \{a_0\}$ and $a_0 \cdot x = 0$ for all $x \in A_3$, $\{c\}$ is obviously an ideal. If N is nonempty, we have

$$c^2 = \Big(\sum_{i,j \in N} \alpha_{ij} \alpha_i \alpha_j \Big) a_0, c^3 = 0,$$

and by the nondegeneracy of $\sum_{i,j \in N} \alpha_{ij} x_i x_j$, we see that $\sum_{i,j \in N} \alpha_{ij} \alpha_i \alpha_j \neq 0$ and hence $\{c\}$ is a two-dimensional subalgebra with c, a_0 as a basis. Since the product of any two elements in A_3 is a multiple of a_0, αa_0, $\alpha \in \Phi$. We see that $\{c\}$ is an ideal. Moreover $A_3^3 = 0$; it is trivially associative. Finally, we shall show that those algebras of type 4) are H-algebras. Let $A = A_1 \oplus A_2 \oplus A_3$ and $d \in A$,

$$d = a + b + c, a \in A_1, b \in A_2, c \in A_3,$$

and c is given by (2). If $a \neq 0$ then $A_1 \subseteq \{d\}$ and $(b + c)^2 = c^2$. Hence $\{b + c\}$ is an ideal and $\{d\} = A_1 + \{b + c\}$ is an ideal. If $a = 0$ then $\{d\} = \{b + c\}$ is again an ideal.

§ 2. Isomorphisms of *H*-algebras

In view of Theorem 1, the problem of isomorphisms of H-algebras reduces to the problem of isomorphisms of H-algebras of third kind. We only consider the finite-dimensional case.

Definition Let (α_{ij}) be a $n \times n$ matrix over Φ. We call it an H-matrix if the associated quadratic form $x(\alpha_{ij})x'$ is nondegenerate where $x = (x_1, \cdots, x_n)$ and x' is the transpose of x. Two H-matrices (α_{ij}) and (β_{ij}) in Φ_n are H-equivalent if there exists $0 \neq \alpha \in \Phi$ and a nonsingular matrix $T \in \Phi_n$ such that

$$(\alpha_{ij}) = \alpha T(\beta_{ij}) T'.$$

It is obvious that H-equivalent is an equivalence relation. Let A be a$(n+1)$-dimensional H-algebra of third type. It is given by a basis $a_0, a_i, i=1, 2, \cdots, n$ and the multiplication table

$$a_0^2 = a_0 a_i = a_i a_0 = 0, a_i a_j = \alpha_{ij} a_0, \alpha_{ij} \in \Phi, \tag{3}$$

for $i, j = 1, 2, \cdots, n$. The structure constants α_{ij} form a matrix of rank n, (α_{ij}), that we will call the structural matrix of A. By Theorem 1, the structural matrix of an $(n+1)$-dimensional H-algebra, A, of third type is an H-matrix of rank n. It is obvious that A is commutative when and only when its structural matrix is symmetric.

Conversely, to a given H-matrix (α_{ij}) of rank n, we may define an $(n+1)$-dimensional H-algebra of third type by (3). This follows from the fact that any sub-quadratic form of a given nondegenerate quadratic form is also nondegenerate. We see also that the algebra so constructed has the given (α_{ij}) as its structural matrix.

Theorem 2 Two $(n+1)$-dimensional H-algebras of third type are isomorphic if and only if their structural matrices are H-equivalent.

Proof Let A, B be two such H-algebras with (α_{ij}), (β_{ij}) their structural matrices and $a_0, a_i, i=1, 2, \cdots, n$, and $b_0, b_i, i=1, 2, \cdots, n$, their standard bases respectively.

Let φ be an isomorphism of A to B. Since all elements in A(or B) with vanishing square are of the form ra_0(or rb_0), $r \in \Phi$, we have

$$a_0\varphi = b_0' = \alpha^{-1} b_0, \ 0 \neq \alpha \in \Phi, \tag{4}$$

$$a_i\varphi = b_i' = \alpha_i b_0 + \sum_{j=1}^{n} t_{ij} b_j, \alpha_i, t_{ij} \in \Phi,$$

$$i = 1, 2, \cdots, n \tag{5}$$

$$b_i' b_j' = \alpha_{ij} b_0'. \tag{6}$$

Furthermore, we may show by direct computation that

$$b_i' b_j' = [\alpha(t_{i1}, t_{i2}, \cdots, t_{in})(\beta_{ij})(t_{j1}, t_{j2}, \cdots, t_{jn})'] b_0'.$$

Hence

$$(\alpha_{ij}) = \alpha T(\beta_{ij}) T', \tag{7}$$

with $T = (t_{ij})$ nondegenerate, for $b_0, b_i', i = 1, 2, \cdots, n$ are linearly

independent.

Conversely, suppose (α_{ij}) and (β_{ij}) have the relation (7), then it is obvious that the linear mapping given by (4), (5) with $\alpha_i \in \Phi, i = 1, 2, \cdots, n$ is an isomorphism. The proof is complete.

In view of Theorem 2, there is a one-to-one correspondence between the class of $(n+1)$-dimensional H-algebras of third type and the equivalence classes of H-equivalent H-matrices. Hence the the problem reduces to a linear algebra problem which we know is unsolved over general field Φ. Yet we have

Theorem 3 Over the real number field, there is a unique commutative $(n+1)$-dimensional H-algebra that corresponds to the unit matrix E_n for $n \geqslant 1$.

Proof Since the structural matrices are symmetric for the commutative case, in the real case, the matrices are equivalent to diagonal matrices with 1, -1 or 0 on the diagonal. Hence any symmetric H-matrix over the real field is equivalent to E or $-E$. But E and $-E$ are H- equivalent and the theorem follows.

Finally, as a corollary of Theorem 2,3, we get Theorem 4. The automorphism group of the unique $(n+1)$-dimensional commutative H-algebra of third type over the real field is isomorphic to the linear group.

$$\begin{pmatrix} \alpha^{-1} & 0 & \cdots & 0 \\ \alpha_1 & & & \\ \vdots & & T & \\ \alpha_n & & & \end{pmatrix},$$

with $0 \neq \alpha, \alpha_i \in \Phi$, $T \in \Phi_n$, $T^{-1} = aT'$.

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代数族上的 Wedderburn 定理

Wedderburn Theorem on Varieties of Algebras

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In this paper we extend a recent result of Liu([6])to a larger class of algebras which includes all well-known algebras. The main theorem here states;if the Wedderburn-Malcev theorem (it is called the Levi- Malcev-Harish- Chandratheorem in Lie algebra over a field of characteristic 0) holds for a certain variety of finite-dimensional algebra, then it holds for the same variety of algebras which are local subideal finite. We do not use the trace argument in this paper as which was heavily depended on in ([6]). Hence the result is independent of the characteristics of the ground field F.

1. Local Solvable Radical

In this paper, we study varieties of algebras which satisfy a certain set of identities (such as associative algebras, Lie algebras, standard algebras, etc.). We do not limit the number of identities. However we do require that

all identities be homogeneous. Those algebras under discussion need not be finite dimensional. But they are locally finite; that is to say, any finite subset X of an algebra A generates a finite-dimensional subalgebra $\langle X \rangle$.

All algebras are considered to be algebras over a fixed field F. Let B be a subalgebra of A. We define $B^{(k)}$ inductively on k by setting $B^{(0)} = B$ and $B^{(k+1)} = B^{(k)} \cdot B^{(k)}$. Thus, for any subalgebra B, we have a descending sequence $B = B^{(0)} \geqslant B^{(1)} \geqslant \cdots \geqslant B^{(n)} \cdots$. We note that, for each i, $B^{(i+1)}$ is an ideal of $B^{(i)}$, hence each $B^{(i)}$ is a subideal of B. A subalgebra B is called a solvable subalgebra if $B^{(k)} = 0$ for some integer k. An algebra is locally solvable if any finite subset of A generates a solvable subalgebra. It is easy to see that solvable algebras are locally solvable and homomorphic images of a locally solvable algebra are locally solvable.

A subalgebra B of A is called a local subideal if, for any finite subset X of A, B is a subideal of $\langle B, X \rangle$, the subalgebra of A generated by B and X, It is clear that any subideal of A is a local subideal of A.

Theorem 1 For any local finite algebra A there exists a unique maximal locally solvable ideal $R = R(A)$. The quotient algebra $\bar{A} = A/R$ is semi-simple. That is A/R contains no non-zero locally solvable ideal.

The proof of this theorem is rather routine. In fact it is exactly the same proof for the existence of the Leviztki radical for associative algebras. Thus we shall omit the proof. As usual we shall call the ideal $R = R(A)$ in Theorem 1 the radical of A. In the sequel, we shall restrict ourself to study varieties of algebras in which solvable radical property for finite-dimensional algebras is weak hereditary. Thus, if B is an ideal of a finite-dimensional algebra A, then $R(B)$ is contained in $R(A)$.

Theorem 2 If a weak hereditary property for finite-dimensional algebras holds in the variety $\mathscr{V}$ and if B is a finite-dimensional local subideal of an algebras A in $\mathscr{V}$, then $R(B)$ is contained in $R(A)$.

Proof If B is semi-simple, then $R(B) = 0$. There is nothing to be shown. We shall assume that B is not semi-simple. Hence B possesses a non-zero solvable ideal C.

Let C_A be the ideal of A generated by C. If $X = \{x_1, x_2, \cdots, x_n\}$ is a subset of C_A, then each x_i is a finite sum of elements of the form $T_{a_1} T_{a_2} \cdots T_{a_k}$, where a_j is in A. T_u is either the right or left multiplication of u. The set Y which consists of all a's appears as factors in some x_i is a finite subset of A. The subalgebra $D = \langle B, Y \rangle$ generated by B and Y is a finite-dimensional subalgebra of A. The algebra B is a subideal of D. This is so because that B is a finite-dimensional local ideal of A, Y is a finite subset of A and A is locally finite. Now we see that C is a solvable ideal of B implies that C is contained in $R(B)$. Hence, by the weak hereditary property, C is contained in $R(D)$. The set, C_D, which is the ideal of D generated by C, is an ideal of $R(D)$. This shows that C_D is a solvable ideal in $R(D)$. Since X is contained in C_D. The subalgebra $\langle X \rangle$ is solvable. This proves that C_A is a locally solvable ideal of A containing C, and that $R(B)$ is contained in $R(A)$.

Corollary If weak hereditary property for finite-dimensional algebras holds in the variety A and if B is a finite-dimensional local subideal of an algebra A, then B is semi-simple if A is.

2. *n*-Varieties

If n is a positive integer greater than 1, we call a (homogeneous) variety $\mathscr{V}$ an n-variety if whenever A is in $\mathscr{V}$ and I is an ideal of A, then I^n is an ideal of A. The set I^n consists of all finite sums of elements of the form $x_1 x_2 x_3 \cdots x_n$, where x_i is in I and the product could be of any association. It is well known that Lie, associative, alternative and (γ, δ)-algebras are 2-algebras while Jordan, standard and Thedy algebras are 3-algebras. We do not know any n-algebra (which is not a k-algebra with $k < n$) for n greater than 3. Thus, in the sequel we shall only study n-varieties for $n = 2$ or 3.

Anderson has shown us (see[1,2]) that if A is a 2-algebra, then for any y, x_1, x_2 in A, there exists 16 constants $\alpha_1, \cdots, \alpha_8, \beta_1, \cdots, \beta_8$ such that

$$y(x_1 x_2) = \alpha_1 (y x_1) x_2 + \alpha_2 (y x_2) x_1 + \alpha_3 x_1 (y x_2)$$

$$+ \alpha_4 x_2(yx_1) + \alpha_5 x_1(yx_2) + \alpha_6 x_2(yx_1) + \alpha_7 x_1(x_2 y) + \alpha_8 x_2(x_1 y); \qquad (1_2)$$

$$(x_1x_2)y = \beta_1(yx_1)x_2 + \beta_2(yx_2)x_1 + \beta_3 x_1(yx_2) + \beta_4 x_2(yx_1) + \beta_5 x_1(yx_2) + \beta_6 x_2(yx_1) + \beta_7 x_1(x_2 y) + \beta_8 x_2(x_1 y). \qquad (2_2)$$

Using the same argument as Anderson, we consider the free A algebra generated by $\{x_1, x_2, x_3, y\}$ and consider the ideal D generated by $\{x_1, x_2, x_3\}$. If it is a 3-variety, then we have D^3 is an ideal of A. Thus we have a similar but more tedious equations for this algebra. If A is a 3-algebra and I is an ideal of A, then for any y, x_1, x_2, x_3 in A there exists 288 constants $\alpha_1, \alpha_2, \cdots, \alpha_{72}, \beta_1, \beta_2, \cdots, \beta_{72}, \gamma_1, \cdots, \gamma_{72}, \delta_1, \cdots, \delta_{72}$ such that

$$\begin{aligned}
y((x_1x_2)x_3) = {} & \alpha_1(yx_1)(x_2x_3) + \alpha_2(yx_1)(x_3x_2) + \alpha_3((yx_1)x_2)x_3 \\
& + \alpha_4((yx_1)x_3)x_2 + \alpha_5 x_2((yx_1)x_3) + \alpha_6 x_3((yx_1)x_2) \\
& + \alpha_7(x_2(yx_1))x_3 + \alpha_8(x_3(yx_1))x_2 + \alpha_9(x_2x_3)(yx_1) \\
& + \alpha_{10}(x_3x_2)(yx_1) + \alpha_{11}x_2(x_3(yx_1)) + \alpha_{12}x_3(x_2(yx_1)) \\
& + \alpha_{13}(x_1y)(x_2x_3) + \alpha_{14}(x_1y)(x_3x_2) + \alpha_{15}((x_1y)x_2)x_3 \\
& + \alpha_{16}((x_1y)x_3)x_2 + \alpha_{17}x_2((x_1y)x_3) + \alpha_{18}x_3((x_1y)x_2) \\
& + \alpha_{19}(x_2(yx_1))x_3 + \alpha_{20}(x_3(x_1y))x_2 + \alpha_{21}(x_2x_3)(x_1y) \\
& + \alpha_{22}(x_3x_2)(x_1y)(x_1y) + \alpha_{23}x_2(x_3(x_1y)) + \alpha_{24}x_3(x_2(x_1y)) \\
& + \alpha_{25}x_1((yx_2)x_3) + \alpha_{26}x_1(x_3(yx_2)) + \alpha_{27}(x_1(yx_2))x_3 \\
& + \alpha_{28}(x_1x_3)(yx_2) + \alpha_{29}(yx_2)(x_1x_3) + \alpha_{30}x_3(x_1(yx_2)) \\
& + \alpha_{31}((yx_2)x_1)x_3 + \alpha_{32}(x_3x_1)(yx_2) + \alpha_{33}((yx_2)x_3)x_1 \\
& + \alpha_{34}(x_3(yx_2))x_1 + \alpha_{35}(yx_2)(x_3x_1) + \alpha_{36}x_3((yx_2)x_1) \\
& + \alpha_{37}x_1((x_2y)x_3) + \alpha_{38}x_1(x_3(x_2y)) + \alpha_{39}(x_1(x_2y))x_3 \\
& + \alpha_{40}(x_1x_3)(x_2y) + \alpha_{41}(x_2y)(x_1x_3) + \alpha_{42}x_3(x_1(x_2y)) \\
& + \alpha_{43}((x_2y)x_1)x_3 + \alpha_{44}(x_3x_1)(x_2y) + \alpha_{45}(x_2y)(x_1x_3) \\
& + \alpha_{46}(x_3(x_2y))x_1 + \alpha_{47}(x_2y)(x_3x_1) + \alpha_{48}x_3((x_2y)x_1) \\
& + \alpha_{49}x_1(x_2(yx_3)) + \alpha_{50}x_1((yx_3)x_2) + \alpha_{51}(x_1x_2)(yx_3) \\
& + \alpha_{52}(x_1(yx_3))x_2 + \alpha_{53}x_2(x_1(yx_3)) + \alpha_{54}(yx_3)(x_1x_2) \\
& + \alpha_{55}(x_2x_1)(yx_3) + \alpha_{56}(x_1(yx_3))x_2 + \alpha_{57}x_2(x_1(yx_3)) \\
& + \alpha_{58}(yx_3)(x_1x_2) + \alpha_{59}x_2((yx_3)x_1) + \alpha_{60}(yx_3)(x_2x_1)
\end{aligned}$$

$$+ \alpha_{61}x_1(x_2(x_3y)) + \alpha_{62}x_1((x_3y)x_2) + \alpha_{63}(x_1x_2)(x_3y)$$
$$+ \alpha_{64}((x_3y)x_1)x_2 + \alpha_{65}x_2(x_1(x_3y)) + \alpha_{66}(x_3y)(x_1x_2)$$
$$+ \alpha_{67}(x_2x_1)(x_3y) + \alpha_{68}((x_3y)x_1)x_2 + \alpha_{69}x_2(x_1(x_3y))$$
$$+ \alpha_{70}((x_3y)x_2)x_1 + \alpha_{71}x_2((x_3y)x_1) + \alpha_{72}((x_3y)x_1)x_2. \tag{1_3}$$

$$y(x_1(x_2x_3)) = \beta_1(yx_1)(x_2x_3) + \cdots + \beta_{72}((x_3y)x_1)x_2. \tag{2_3}$$

$$((x_1x_2)x_3)y = \gamma_1(yx_1)(x_2x_3) + \cdots + \gamma_{72}((x_3y)x_1)x_2. \tag{3_3}$$

$$(x_1(x_2x_3))y = \delta_1(yx_1)(x_2x_3) + \cdots + \delta_{72}((x_3y)x_1)x_2. \tag{4_3}$$

From these equations we easily see:

Lemma A If A is a 3-algebra and B is an ideal of A, then $B^2 + B^2A + AB^2$ is a 2-sided ideal of A.

Corollary If A is a commutative 3-algebra and B is an ideal of A, then $B^2 + B^2A$ is an ideal of A.

Note For Jordan algebra A (which is a 3-algebra) Penico has shown that $B^2 + B^2A$ is an ieal of A if B itself is an ideal of A.

Theorem 3 If A is a simple (which means A possesses no non-trivial ideal and $A^2 \neq 0$) algebra which has a finite-dimensional local subideal B, then A is contained in B. Hence A is finite dimensional.

Proof Since A is semi-simple, by Theorem 2, B is a finite-dimensional semi-simple subalgebra of A. We can find a non-zero subideal $C = B^{(k)}$ for some integer k such that $C = C^2 = C^3$.

For any arbitrary x in A, C is a subideal of $\langle B, X\rangle$. Because C is a subideal of B and B is a subideal of $\langle B, x\rangle$. Thus, from those identities for n-algebras ((1_2), (2_2) for 2-algebras; (1_3), (2_3), (3_3) and (4_3) for 3-algebras) we check that C is indeed an ideal of $\langle B, X\rangle$. But this implies $Cx \leqslant C$ and $xC \leqslant C$. Hence C is an ideal of A.

The algebra A is simple and C is a non-zero ideal of A. Thus $A = C$ and is contained in B.

Theorem 4 If A is a semi-simple algebra which has a finite-dimensional local subideal B, suppose that B has a minimal ideal C, then C is an ideal of A.

Proof Since C is a minimal ideal of B, we conclude that $C = C^2$ if A is a 2-algebra and $C = C^3$ if A is a 3-algebra. The rest of the proof that C is an ideal of A is same as that in last theorem.

3. Local System

A local system (see [3]) of an algebra A is a collection $\{A_\lambda / \lambda \in \Lambda\}$ of subalgebras of A such that (1) $\bigcup A_\lambda = A$ and (2) for λ, μ in Λ there exists a σ in Λ such that $\langle A_\lambda, A_\mu \rangle \leqslant A_\sigma$. An algebra A is said to be local subideal finite if it has a local system $\{A_\lambda / \lambda \in \Lambda\}$ such that (1) $\dim_F(A_\lambda) < \infty$ and (2) for each λ, A_λ is a local subideal of A.

Theorem 5 *Let* $\mathscr{V}$ be an n-variety. Assume that in this variety any finite-dimensional algebra is semi-simple if and only if it is a direct sum of simple algebras and solvable radical property is weak hereditary. Then a local subideal finite algebra A in $\mathscr{V}$ is semi-simple if and only if it is a direct sum of simple finite-dimensional algebras.

Proof Assume that A is semi-simple. Then, by Theorem 2, each A_λ is semi-simple. But $\dim_F A_\lambda < \infty$; thus, by the assumption, for each λ, A_λ is a direct sum of finite-dimensional simple subalgebras $A_\lambda = \sum_i \oplus A_{\lambda,i}$. Moreover, by Theorem 4, each $A_{\lambda,i}$ is an ideal of A. Thus $A = \sum_{\lambda,i} A_{\lambda,i}$ and by removing duplicated simple components from this sum, $A = \sum \oplus A_{\lambda,i}$.

Theorem 6 Any homomorphic image $\bar{A}$ of a local subideal finite algebra A is a local subideal finite algebra. Any subalgebra A' of a local subideal finite algebra A is a local subideal finite algebra.

Proof If $\{A_\lambda : \lambda \in \Lambda\}$ is a local subideal system of A, then it is easy to verify that $\{\bar{A}_\lambda : \lambda \in \Lambda\}$ is a local subideal system of $\bar{A}$.

If A' is a subalgebra of A, then for each $\lambda \in \Lambda$, we let $A_\lambda' = A' \cap A_\lambda$. Then it is easy to see that $\{A_\lambda' \mid \lambda \in \Lambda\}$ is a local system of A. Moreover dim A' is finite for each λ. If X is a finite subset of A', then, for each $\lambda \in \Lambda$, there is a chain of subideals $A_\lambda \lhd C_1 \lhd C_2 \lhd \cdots \lhd C_r =$

$\langle A_\lambda, X\rangle$. Thus, we have $A' \cap A_\lambda \lhd A' \cap C_1 \lhd A' \cap C_2 \lhd \cdots \lhd A' \cap C_r = A' \cap \langle A_\lambda, X\rangle$. But $\langle A' \cap A_\lambda, X\rangle \leqslant A' \cap \langle A_\lambda, X\rangle$. So $A' \cap A_\lambda = A' \cap A_\lambda \cap \langle A' \cap A_\lambda, X\rangle \lhd A' \cap C_1 \cap \langle A' \cap A_\lambda, X\rangle \lhd \cdots \lhd A' \cap \langle A_\lambda, X\rangle \cap \langle A' \cap A_\lambda, X\rangle = \langle A' \cap A_\lambda, X\rangle$. This shows that A_λ' is a subideal of $\langle A_\lambda', X\rangle$ and that A' is a local subideal finite algebra.

4. Wedderburn-Malcev-type Theorems

If A is an algebra, then an automorphism is called an inner automorphism if $\Phi = 1 + f(R_{a_i}, L_{b_j})$, where a_i, b_j are elements of A and at least one of them is in R, the radical of A. R_u (respectively L_v) is the right (respectively left) multiplication of u (respectively v) and $f(x_i, y_j)$ is a polynomial in x_i, x_j (non-commutative) without constant term. We also require an inner automorphism Φ to satisfy condition (0): Whenever B is a finite-dimensional algebra in the variety $\mathscr{V}$ containing. A, and $R(A) \subseteq R(B)$, then any inner automorphism Φ of A is an automorphism of B.

Inner automorphism of this type are plenty among all well-known varieties of algebras. (See [4].) Here we note:

Lemma B If A is a local finite algebra in $\mathscr{V}$ and B is a finite-dimensional local subideal of A, then any inner automorphism of B is an inner automorphism of A.

Proof Let x, y be elements of A and $D = \langle B, x, y\rangle$ be the subalgebra generated by B, x and y. Then B is a subideal of D and $\dim_F D$ is finite. So Φ is an inner automorphism of D. This shows that Φ is in fact an automorphism of A.

An n-variety $\mathscr{V}$ is said to be of WM-type if for any finite-dimensional algebra A in $\mathscr{V}$ we have (1) $A = S \oplus R$, where $R = R(A)$ is the solvable radical of A and the sum in the above equation is semi-direct, (2) A is a direct sum o simple subalgebra, i.e., $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$, if A is semi-simple, (3) if S_1 is a semi-simple subalgebra of A, then there exists an inner automorphism Φ such that $S_1\Phi \subseteq S$, and (4) the solvable radical property for finite-dimensional algebras is weak hereditary.

Associative algebras, Lie algebras over fields of characteristic zero, Jordan algebras and alternative algebras are well known to be algebras of WM-type.

Theorem 7 If $\mathscr{V}$ is a 2-variety of WM-type, then for any local subideal finite algebra A in $\mathscr{V}$ there exists a semi-simple subalgebra S that $A = S \oplus R$, where $R = R(A)$ is the local solvable radical of A.

Proof Let $R = R(A)$ be the radical of A. Then $\bar{A} = A/R$ is semi-simple. By Theorem 6, the algebra $\bar{A}$ is local subideal finite. Thus, $\bar{A} = \oplus \sum_{\lambda \in W} \bar{A}$ is a direct sum of simple subalgebras $\bar{A}_\lambda$. Each $\bar{A}_\lambda$ is of finite-dimension but W may be infinite. We first put an well-ordering on W and then we shall show that we can, for each λ in W, pick a simple subalgebra P_λ in A such that P_λ is of finite dimension, $P_\lambda \cap R = 0$, and $\bar{P}_\lambda = \bar{A}_\lambda$. Moreover $P_\lambda \cdot \oplus \sum_{\sigma<\lambda} P_\sigma \cdot P_\lambda$, so $S = \oplus \sum_{\lambda \in W} P_\lambda$ is a semi-simple subalgeba of A such that $A = S \oplus R$ (semi-direct sum).

Let $\{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_m\}$ be a basis of $\bar{A}_1$. For each i, let $\bar{a}_i = a_i + R$, where a_i is in A. Pick member B of the local system of A which contains a_i. Since $\bar{A}_1 \subseteq \bar{B}, \bar{A}_1 = \bar{A}_1^{(k)} \subseteq \bar{B}^{(k)}$, the algebra B is not solvable. If the chain $B \supseteq B^{(1)} \supseteq B^{(2)} \cdots B^{(k)} = B^{(k+1)} = \cdots$ stabilizes at $B^{(k)=C}$, then C is a non-zero idemportent subalgebra of A, i.e., $C^2 = C$.

We know that C is a subideal of B and B, in turn, is a local subideal of A. So C is an ideal of A (recall that $C = C^2 = C^3 = \cdots$). In summary, we have found a finite-dimensional ideal C of A such that $\bar{A}_1 \subseteq \bar{C} \subseteq \bar{A}$. The algebra $\bar{C}$ is a finite-dimensional semi-simple subalgebra of A, and the radical R (C) of C is $C \cap R$. Thus we have a semi-simple subalgebra P of C such that $C = R(C) \oplus P$ (semi-direct sum). Therefore there is a subalgebra P_1 of P such that $\bar{P}_1 = \bar{A}_1$ and $P_1 \cap R = P_1 \cap C \cap R = 0$.

If m is a finite ordinal, and for each $r < m$, we have already chosen a finite-dimensional simple subalgebra P_r of A such that $\bar{P}_r = \bar{A}_r, P_r \cap A = 0$, and $H = P_1 \oplus \cdots \oplus P_{m-1}$, we shall find P_m, a finite-dimensional simple subalgebra of A, such that $P_m \cap R = 0, \bar{P}_m = \bar{A}_m$, and $P = P_1 \oplus \cdots \oplus P_{m-1}$

$\oplus P_m$ is a direct sum.

Let $\{\bar{b}_1, \bar{b}_2, \cdots, \bar{b}_n\}$ be a basis of $\bar{A}_m$. If, for each i, let $\bar{b}_i = b_i + R$ and B is a member in the local system of A containing all b's. Then as we have shown above, B is a non-solvable subalgebra of A and there exists a non-zero idempotent ideal C of A such that $\bar{A}_m \subseteq \bar{C} \subseteq \bar{A}$. The algebra $\bar{C}$ is a finite-dimensional semi-simple subalgebra of $\bar{A}$ and the radical $R\ (C)$ of C is $R\ (C) = C \cap R$. Thus, we have a semi-simple subalgebra Q of C such that $C = R(C) \oplus Q$ (semi-direct sum). Therefore, Q contains a subalgebra Q_m such that $\bar{Q}_m = \bar{A}_m$ and $Q_m \cap R = Q_m \cap C \cap R = 0$.

If $HQ_m = Q_m H = 0$, then $P = P_1 \oplus \cdots \oplus P_{m-1} \oplus Q_m$ is a finite-dimensional subalgebra which we are looking for. Suppose, this is not the case: we may find a member B in the local system of A which contains $Q_m, P_1, \cdots, P_{m-1}$. Again, B contains a finite-dimensional idempotent ideal D and $D = R(D) + S$, where S is a semi-simple subalgebra of D. Because that $\bar{D}$ contains $\bar{A}_1 \oplus \bar{A}_2 \oplus \cdots \oplus \bar{A}_{m-1} \oplus \bar{A}_m$, we have $S_1, S_2, \cdots, S_m$, each of this is a simple subalgebra of D such that $\bar{S}_i = \bar{A}_i$, $S_i \cap R = 0$, and $P = S_1 \oplus \cdots \oplus S_m$ is a direct summand of S.

If C is a finite-dimensional ideal of an algebra A and H is a subalgebra of A, then $R_H = \{x \text{ in } H/Cx = 0\}$ and $L_H = \{x \text{ in } H/xC = 0\}$ are subspaces of H each of finite co-dimension. Let $K = R_H \cap L_H$. Then K is a subspace of H of finite co-dimension. Thus we have $H = K \oplus K^*$, where the sum is the direct sum of vector spaces and K^* is of finite dimension.

Now we are ready to show the theorem by transfinite induction. If σ is an ordinal such that, for each $\lambda < \sigma$, there exists a subalgebra P_λ of A_λ such that $\bar{P}_\lambda = \bar{A}_\lambda, P_\lambda \cap R = 0$ and $H = \sum_{\lambda<\sigma} \oplus P_\lambda$ (direct sum of subalgebras). Then as we have shown for $\bar{A}_1$ there exists a finite-dimensional ideal C_σ of A such that $C_\sigma^2 = C_\sigma, \bar{C}_\sigma = \bar{A}_\sigma, C_\sigma = (C_\sigma \cap R) \oplus P'$ (semi-direct sum), $P' = P'_{\sigma,1} \oplus P'_{\sigma,2} \oplus \cdots \oplus P'_{\sigma,r}$ where, in particular, we let $\bar{P}'_{\sigma,1} = \bar{A}_\sigma$. By the last paragraph, the subalgebra H has the decomposition $H = K \oplus K^*$. But A is a 2-variety and C is an ideal of

A, we see easily that K is an ideal of H. Thus, the $H = K \oplus K^*$ is a direct sum of ideals, noting that H is a direct sum of simple subalgebras. Moreover, $K^* = P_1 \oplus \cdots \oplus P_k$, by reindexing.

Let B be an member of the local system of A containing $C, P_1, \cdots, P_k$. Since B contains a finite-dimensional idempotent ideal D. Moreover $D = R(D) + P''$, where P'' is a semi-simple subalgebra of D, and $P'' = P_1'' \oplus P_2'' \oplus \cdots \oplus P_k'' \oplus P_\sigma'' \oplus P_\beta'' \oplus \cdots \oplus P_\alpha''$, where $\overline{P}_i'' = \overline{P}_i = \overline{A}_i$, and $\overline{P}_\sigma'' = \overline{P}_\sigma' = \overline{A}_\sigma$. Since K^* is a semi-simple subalgebra of D, there exists an inner automorphism Φ such that $K^* \Phi^{-1} \subseteq P''$. Thus after rearranging the index $P_i''\Phi = P_i$ for $i = 1, 2, \cdots, k$. Let $P = P''\Phi = P_1''\Phi \oplus P_2''\Phi \oplus \cdots \oplus P_k''\Phi \oplus P_\sigma''\Phi \oplus P_\beta''\Phi \oplus \cdots \oplus P_\alpha''\Phi = P_1 \oplus P_2 \oplus \cdots \oplus P_k \oplus P_\sigma \oplus P''_\beta{}''\Phi \oplus \cdots \oplus P_\delta''\Phi$, where $P_\sigma = P_\sigma''\Phi$. We see that $\overline{P}_\sigma'' = \overline{P}_\sigma = \overline{A}_\sigma$, P_σ is semi-simple, $P_\sigma \cap R = 0$, and $P_\sigma \cdot P_i = P_i \cdot P_\sigma = 0$ for $i = 1, 2, \cdots, k$. On the other hand, recognizing the expression formula of the inner automorphism Φ we have $P_\sigma = P_\sigma''\Phi$ is contained in the ideal C which contains P_σ''. Hence P_σ is contained in the ideal C which contains P_σ''. Hence P_σ is contained in C and so $P_\sigma \cdot K = K \cdot P_\sigma = 0$. Thus, $H \cdot P_\sigma = P_\sigma \cdot H = 0$. So $H^* = H + P$ is the direct sum $\sum_{\lambda \leqslant \sigma} \oplus P_\lambda$ of simple subalgebras such that $\overline{P}_\lambda = \overline{A}_\lambda$ for $\lambda \leqslant \sigma$. This completes the induction proof.

Remark We would like to see Theorem 7 holds true for 3-algebras. However, there is a little hurdle we do not know how to overcome.

In the transfinite induction part of the proof of Theorem 7, the annihilator K of the ideal C is indeed an ideal of A, if A is a 2-algebra. We are not sure this is the case for 3-algebras. Thus, we shall say a 3-variety $\mathscr{V}$ satisfies condition (T) if whenever A is in $\mathscr{V}$ and I is an ideal of A such that $I^2 = I$, then $K = \operatorname{Ann} C = \{x \in A \mid xC = Cx = 0\}$ is an ideal of A. We see that this condition holds for most well-known varieties. For example the identity $(xy)(uv) = -(x \cdot u \cdot (y \cdot v) - (x \cdot v) \cdot (y \cdot u) + [(x \cdot u) \cdot y] \cdot v + [(x \cdot v) \cdot y] \cdot u + [(u \cdot v) \cdot y] \cdot x$ for Jordan algebra yields condition (T). (Let u, v be taken from C and y from K.) So the variety $\mathscr{V}$

of Jordan algebras is a 3-variety satisfies (T).

Theorem 8 If $\mathscr{V}$ is a 3-variety of WM-type satisfying condition (T), then for any local subideal finite algebra A in $\mathscr{V}$ there exists a semi-simple subalgebra S such that $A = S \oplus R$, where $R = R(A)$ is the local solvable radical of A.

Theorem 9 If $\mathscr{V}$ is a n-variety of WM-type and A is a local subideal finite algebra in $\mathscr{V}$ and $A = R \oplus S$ is a Wedderburn decomposition of A. Then for any finite-dimensional semi-simple subalgebra Q of A, there exists an inner automorphism of the form $\Phi = 1 + f(R_{a_i}, L_{b_j})$ such that $Q\Phi \subseteq S$.

Proof Since $\bar{Q} \subseteq \bar{A} = \bar{S} = \oplus \sum_\lambda \bar{P}_\lambda$, where $S = \oplus \sum_\lambda P_\lambda$ and $\dim_F Q < \infty$ by Theorem 5, we may choose $P_1, P_2, \cdots, P_n$ from the direct sum of S such that $\bar{Q}$ is contained in $\bar{P}_1 \oplus \bar{P}_2 \oplus \cdots \oplus \bar{P}_n$.

Let B be a member of the local system of A which contains $Q, P_1, \cdots, P_n$. Because B is not solvable, there exists an integer k such that $B^{(k)} = B^{(k+1)} = \cdots$. Again, we see that $C = B^{(k)}$ is a finite-dimension ideal of A containing $Q, P_1, \cdots, P_n$. By the finite dimensionality, $C = R(C) \oplus S^*$, where S^* is a semi-simple subalgebra of C containing $P_1 \oplus P_2 \oplus \cdots \oplus P_n$. So we have an inner automorphism ϕ of C (hence of A) such that $Q\phi \subseteq S^*$. But $\bar{Q}$ is contained in $\bar{P}_1 \oplus \cdots \oplus \bar{P}_n$. Therefore, $Q\Phi \subseteq P_1 \oplus \cdots \oplus P_n$. This completes the proof.

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二 结合代数

II. Associative Algebras

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路代数的同构①

Isomorphisms of Path Algebras

刘绍学　罗运纶　肖　杰

摘　要　设 $K(\Delta)$ 表示有向图 Δ 在域 K 上的路代数,本文证明:(1)$K(\Delta)^+$ 与 $K(\Gamma)^+$ 代数同构当且仅当 Δ 与 Γ 边同构.(2)$K(\Delta)$ 与 $K(\Gamma)$ 代数同构当且仅当 Δ 与 Γ 同构.

一

一个有向图是 $\Delta=(\Delta_0,\Delta_1,s,e)$,$\Delta_0$ 和 Δ_1 是两个集合,s,e 是从 Δ_1 到 Δ_0 的映射.称 Δ_0 是 Δ 的顶点集,Δ_1 是 Δ 的箭向集.对 $\alpha\in\Delta_1$,$s(\alpha)$,$e(\alpha)$ 分别叫做 α 的起点和终点,写为 $a\xrightarrow{\alpha}b$,$a=s(\alpha)$,$b=e(\alpha)$,并且,当 $a\neq b$ 时,称 a,b 互为邻点.

有向图 Δ 中从 a 到 b 的长度 $\geqslant 1$ 的道路是 $(a\mid\alpha_1,\cdots,\alpha_l\mid b)$:对所有 $1\leqslant i\leqslant l$,$e(\alpha_i)=s(\alpha_{i+1})$,$a$ 是 α_1 的起点,b 是 α_l 的终点.另外,对 Δ 的顶点 a,我们将它看成长度为零的道路,记为 $(a\mid a)$.当然,也将箭向看成长度为 1 的道路.一个从 a 到 a 的长度 $\geqslant 1$ 的道路称为循环,特别,称从 a 到 a 的箭向为圈.

对于固定域 K,Δ 的路代数 $K(\Delta)$:首先是由 Δ 中所有道路为基作成的 K-向量空间,乘法由道路之间的乘法按分配律给出,两个道路相乘定义为:

① 本文 1985 年 11 月 27 日收到.

$$(a \mid \alpha_1, \cdots, \alpha_l \mid b)(c \mid \beta_1, \cdots, \beta_m \mid d) = \begin{cases} (a \mid \alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_m \mid d), & b = c \\ 0, & b \neq c \end{cases}$$

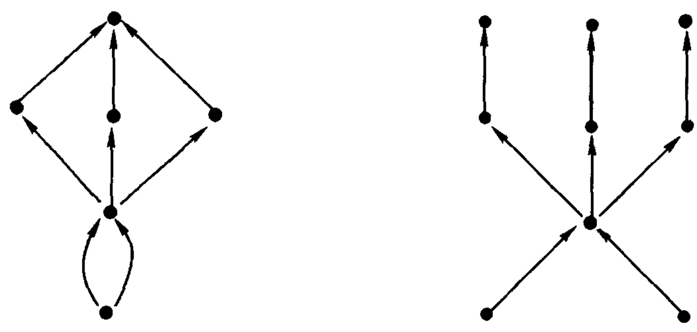
按这种方式我们得到结合代数 $K(\Delta)$. $K(\Delta)$ 有单位元当且仅当 Δ_0 是有限的,其单位元等于 $\sum\limits_{x \in \Delta_0} (x \mid x)$. $K(\Delta)$ 是有限维 K-代数当且仅当 Δ_0, Δ_1 是有限的,且 Δ 中没有循环道路. $K(\Delta)^+$ 表示 $K(\Delta)$ 中所有箭向生成的子代数,这也是 $K(\Delta)$ 的理想.

二

本节旨在证明:若有代数同构 $K(\Gamma)^+ \simeq K(\Delta)^+$,则 Γ 与 Δ 边同构.

设 $\Gamma = (\Gamma_0, \Gamma_1)$ 和 $\Delta = (\Delta_0, \Delta_1)$ 是两个有向图,称 Γ 和 Δ 是边同构的,如果存在双射 $\varphi: \Gamma_1 \to \Delta_1$,使对 $\alpha_1, \alpha_2 \in \Gamma_1$, α_1 连 α_2(即 α_1 的终点是 α_2 的起点)当且仅当 $\varphi(\alpha_1)$ 连 $\varphi(\alpha_2)$. 显然,Γ 和 Δ 边同构则 $K(\Gamma)^+$ 和 $K(\Delta)^+$ 代数同构.

$a \in \Gamma_0$ 称为是 Γ 的孤立点,如果 a 既不是起点又不是终点;称 a 是汇点,如果 a 只能是终点不是起点;称 a 是源点,如果 a 只能是起点不能是终点. 如果 Γ 的每个汇点和源点的邻点都是唯一的,则称 Γ 是不可裂图. 显然任何一个有向图都与一个不可裂图边同构.



Γ 及其不可裂图

引理 2.1 如果两个不可裂图是边同构的,则这两个不可裂图是同构的.

证明:设 $\varphi: \Gamma = (\Gamma_0, \Gamma_1) \to \Delta = (\Delta_0, \Delta_1)$ 是边同构,令 $\varphi: \Gamma_0 \to \Delta_0$ 是如下对应:任意 $a \in \Gamma_0$,因不可裂图没有孤立点,那么 a 是箭向 α 的起点

或终点,若 a 是α 的起点,则令 $\varphi(a)$ 是 $\varphi(\alpha)$ 的起点;若 a 是α 的终点,则令 $\varphi(a)$ 是 $\varphi(\alpha)$ 的终点.这个定义是合理的,若 a 是α_1 的终点,又是 α_2 的起点,那么 α_1 连 α_2,所以 $\varphi(\alpha_1)$ 连 $\varphi(\alpha_2)$;若 a 是α_2,α_3 的起点,必然有 α_1 以 a 为终点,那么 $\varphi(\alpha_1)$ 的终点等于 $\varphi(\alpha_2)$, $\varphi(\alpha_3)$ 的起点. $\varphi:\Gamma_0\to\Delta_0$ 是双射,因为如上导出的 $\varphi^{-1}:\Delta_0\to\Gamma_0$ 显然满足 $\varphi\varphi^{-1}=1_{\Delta_0}$, $\varphi^{-1}\varphi=1_{\Gamma_0}$.若从 a 到b 的箭向有$\{\alpha_i\}_{i\in I}$,则从 $\varphi(a)$ 到 $\varphi(b)$ 的箭向有$\{\varphi(\alpha_i)\}_{i\in I}$.

由此知,每个有向图都有唯一的不可裂图与其边同构.

对于 $K(\Gamma)^+$,有自然的 Z-分次 $K(\Gamma)^+=K(\Gamma)^+(1)\oplus\cdots\oplus K(\Gamma)^+(n)\oplus\cdots$,其中 $K(\Gamma)^+(n)$ 是 Γ 中长度等于n 的道路的线性和.任一元素 $x\in K(\Gamma)^+$,可唯一表为 $x=x(1)+x(2)+\cdots$(有限和).其中 $x(n)\in K(\Gamma)^+(n)$.

代数同构 $f:K(\Gamma)^+\to K(\Delta)^+$ 称为是保次的,如果对任意 $n\geqslant 1$, $f(K(\Gamma)^+(n))=K(\Delta)^+(n)$,即,将 Γ 中的箭向映成 Δ 中箭向的线性和.

引理2.2 如果有代数同构$f:K(\Gamma)^+\to K(\Delta)^+$,则存在保次的代数同构 $f_1:K(\Gamma)^+\to K(\Delta)^+$.

证明:设 $f:K(\Gamma)^+\to K(\Delta)^+$ 是代数同构.对于 $K(\Gamma)^+$ 中一条道路 $(a\mid\alpha_1\cdots\alpha_l\mid b)$,令 $f_1(a\mid\alpha_1,\cdots,\alpha_l\mid b)=f(a\mid\alpha_1,\cdots,\alpha_l\mid b)(l)$,显然,$f_1(a\mid\alpha_1\cdots\alpha_l\mid b)=f(\alpha_1)(1)\cdots f(\alpha_l)(1)$,所以 f_1 线性扩张到 $K(\Gamma)^+$ 上是$K(\Gamma)^+$到$K(\Delta)^+$ 的代数同态.按上述方法由f^{-1}得到的f_1^{-1}是$K(\Delta)^+$ 到 $K(\Gamma)^+$ 的保次代数同构,由于对任意 $n\geqslant 1$, $f(K(\Gamma)^+(n))\leqslant K(\Delta)^+(n)\oplus K(\Delta)^+(n+1)\oplus\cdots$,易知 $f_1f_1^{-1}=1_{K(\Delta)^+}$, $f_1^{-1}f_1=1_{K(\Gamma)^+}$.

起点是源点的箭向(道路)称为是起箭向(起道路);终点是汇点的箭向(道路)称为是终箭向(终道路);起点是源点、终点是汇点的箭向(道路)称为是孤立箭向(孤立道路);既不是起箭向(起道路)又不是终箭向(终道路)的箭向(道路)叫做中间箭向(中间道路).

$L(\ \ )$ 和 $R(\ \ )$ 分别表示在 $K(\Gamma)^+$ 或 $K(\Delta)^+$ 中的左零化子和右零化子.

引理2.3 设 $f:K(\Gamma)^+\to K(\Delta)^+$ 是保次的代数同构,则

1)f 将起箭向映成起箭向的线性和.

2)f 将终箭向映成终箭向的线性和.

3)f 将孤立箭向映成孤立箭向的线性和.

证明: 只证明1),设 α 是 Γ 的起箭向,$f(\alpha)=k_1\beta_1+k_2\beta_2+\cdots+k_n\beta_n$,$0\neq k_i\in K$,$L(\alpha)=K(\Gamma)^+$,$L(f(\alpha))=\bigcap_{i=1}^{n}L(\beta_i)$,而 f 是同构.所以 $K(\Delta)^+=\bigcap_{i=1}^{n}L(\beta_i)$,所以 $L(\beta_i)=K(\Delta)^+$.$1\leqslant i\leqslant n$,即 β_i 是起箭向.

引理 2.4　1) 设 α_1,α_2 是 Γ 的两个箭向(道路).如果 $L(\alpha_1)\leqslant L(\alpha_2)$,则 α_2 是起箭向(起道路).

2) 设 α_1,α_2 是 Γ 的两个箭向(道路),如果 $R(\alpha_1)\leqslant R(\alpha_2)$,则 α_2 是终箭向(终道路).

证明: 只证明1),因为 $L(\alpha_1)\neq L(\alpha_2)$,所以 α_1 与 α_2 起点不同,如果有 α 连α_2,那么 α 不连α_1,$\alpha\in L(\alpha_1)$,$\alpha\notin L(\alpha_2)$,这与 $L(\alpha_1)\leqslant L(\alpha_2)$ 矛盾.故 α_2 是起箭向.

引理 2.5　设 $f:K(\Gamma)^+\to K(\Delta)^+$ 是保次的代数同构,则

1)$f(A)=B_1+B_2$,A 是Γ 中任意具有相同终点的起箭向的线性和,B_1 是 Δ 中具有相同终点的起箭向的线性和,B_2 是 Δ 中孤立箭向的线性和;$A\neq 0$ 则 $B_1\neq 0$.

2)$f(A)=B_1+B_2$,A 是Γ 中具有共同起点的终箭向的任一个线性和,B_1 是 Δ 中具有相同起点的终箭向的线性和,B_2 是 Δ 中孤立箭向的线性和;$A\neq 0$ 则 $B_1\neq 0$.

3)$f(A)=B_1+B_2+B_3+B_4$,A 是Γ 中具有相同起点和相同终点的中间箭向的任意一个线性和,B_1 是 Δ 中具有相同起点和相同终点的中间箭向的一个线性和,B_2 是与 B_1 中箭向有相同终点的起箭向的一个线性和,B_3 是与 B_1 中箭向有相同起点的终箭向的一个线性和,B_4 是孤立箭向的一个线性和;若 $A\neq 0$,则 $B_1\neq 0$.

证明: 只证明3),设 $\alpha_1,\cdots,\alpha_n$ 是Γ 中具有相同起点和相同终点的中间箭向.$0\neq k_i\in K$,设 $f(k_1\alpha_1+\cdots+k_n\alpha_n)=l_1\beta_1+l_2\beta_2+\cdots+l_m\beta_m$,$\beta_i$ 是 Δ 中的不同箭向,$0\neq l_i\in K$,由 f 是同构知$f(L(\alpha_1))=\bigcap_{i=1}^{m}L(\beta_i)$,$f(R(\alpha_1))\leqslant\bigcap_{i=1}^{m}R(\beta_2)$.不妨设 $f^{-1}(\beta_1)$ 的展开式中出现 α_1 的非零项,特别有 $f^{-1}(L(\beta_1))\leqslant L(\alpha_1)$,$f^{-1}(R(\beta_1))\leqslant R(\alpha_1)$,从而有 $f(L(\alpha_1))=$

$L(\beta_1), f(R(\alpha_1)) = R(\beta_1)$.那么 $R(\beta_1) \leqslant R(\beta_i), L(\beta_1) \leqslant L(\beta_i)$. $i = 1, \cdots, m$,由引理2.4知3)成立.

注.引理2.3、2.4在不假定 f 是保次时,将"箭向"改为"道路",结论也成立.

定理 2.1 设 Γ 和 Δ 是任意有向图,如果有 K-代数同构 f: $K(\Gamma)^+ \to K(\Delta)^+$,则 Γ 和 Δ 边同构.

证明:由引理2.2,可设 f 是保次的代数同构.设 Γ 的孤立箭向全体张成的向量空间为 V_1,Δ 的孤立箭向全体张成的向量空间为 W_1,由引理2.3知,f 和 f^{-1} 诱导了 V_1 和 W_1 之间的 K-同构(作为向量空间).这样,存在 Γ 的孤立箭向集到 Δ 的孤立箭向集之间的一一对应 φ.对于 Γ 的不是孤立箭向的起箭向 α,设 V_2 是与 α 有相同终点的起箭向全体张成的向量空间,由引理2.5,$f(\alpha) = B_1 + B_2$,B_1 是具有相同终点 b 的起箭向(不是孤立箭向)的线性和,B_2 是孤立箭向的线性和.设以 b 为终点的起箭向全体张成向量空间 W_2,由引理2.5知,f 和 f^{-1} 诱导了 V_2 和 W_2 之间的 K-同构.这样,存在 Γ 的以a 为终点的起箭向集到 Δ 的以 b 为终点的起箭向集的一一对应 φ.对于 Γ 的不是孤立箭向的终箭向,与其有相同起点的终箭向全体张成向量空间 V_3,由引理2.5,存在具有相同起点的终箭向(不是孤立箭向)全体张成的空间 W_3,f 与 f^{-1} 诱导了 V_3 和 W_3 之间的 K-同构;这样,有 V_3 中出现的终箭向全体到 W_3 中出现的终箭向全体的一一对应 φ.对于 Γ 的中间箭向α,设 V_4 是 Γ 中与α 具有相同起点和相同终点的中间箭向全体张成的空间,由引理2.5知,$f(\alpha) = B_1 + B_2 + B_3 + B_4$,$B_1 \neq 0$,$B_1$ 中出现中间箭向 β,Δ 的与 β 相同起点和相同终点的中间箭向全体张成空间 W_4,f 和 f^{-1} 诱导了 V_4 和 W_4 之间的 K-同构.因而可构造 V_4 中出现的中间箭向和 W_4 中出现的中间箭向之间的一一对应.最后,我们得到 Γ 的箭向集Γ_1 到 Δ 的箭向集 Δ_1 的一一对应 φ,从引理2.5看出 φ 诱导了边同构.

三

这一节旨在证明:若有 K-代数同构$K(\Gamma) \simeq K(\Delta)$,则 Γ 与 Δ 同构.

称 Γ 中以a 为顶点的圈是孤立圈,如果 a 在Γ 中没有邻点.Γ 中孤立点组成的子图记为 Γ',Γ 中孤立圈组成的子图记为Γ'',这样,$\Gamma = \Gamma' \vee$

$\Gamma'' \vee \Gamma'''$(图的联). Γ''' 的任一顶点总是 Γ 中一个不是圈的箭向的起点或终点. Γ 中由所有不是圈的箭向所组成的子图记为 Γ^*, $K(\Gamma^*)^+$ 在 $K(\Gamma)$ 中生成的理想记为 $\langle K(\Gamma^*)^+\rangle$.

引理 3.1 若有代数同构 $f: K(\Gamma) \to K(\Delta)$, 则 $f(\langle K(\Gamma^*)^+\rangle) = \langle K(\Delta^*)^+\rangle$.

证明: 只要证明 $f(K(\Gamma^*)^+) \subseteq \langle K(\Delta^*)^+\rangle$. 对于 Γ 中不是圈的箭向 α, 设 $f(\alpha) = \sum_i k_i b_i + B$, $k_i \in K$, b_i 是 Δ 的顶点, B 是 Δ 中长度 $\geqslant 1$ 的道路的线性和, 因为 $\alpha^2 = 0$, 所以 $f(\alpha)^2 = 0$, 特别有 $\left(\sum_i k_i b_i\right)^2 = 0$, 那么 $\sum_i k_i{}^2 b_i = 0$, $k_i = 0$; 因而 $f(\alpha) = B$, 由于 $f(\alpha)^2 = 0$, B 的展开式中不能出现循环项.

引理 3.2 设 $f: K(\Gamma) \to K(\Delta)$ 是代数同构, 则 $f(K(\Gamma')) = K(\Delta')$.

证明: 设 a_1 是 Γ 的孤立点, 因为 $f(a_1)^2 = f(a_1)$, 由此知 $f(a_1)$ 的线性展开式中不能出现循环项. 如果 $f(a_1)$ 的展开式中有一项是长度 $\geqslant 1$ 的道路 x, x 的起点是 b, 终点是 b'. $b \neq b'$. 那么 $0 \neq bf(a_1)b' \in \langle K(\Delta^*)^+\rangle$, 由引理3.1知, $0 \neq f^{-1}(b)a_1 f^{-1}(b') \in \langle K(\Gamma^*)^+\rangle$, 矛盾. 所以 $f(a_1)$ 是 Δ 的顶点的线性和. 若 $f(a_1)$ 的展开式中出现非孤立点, 存在 Δ 的箭向 β 使 $f(a_1)\beta \neq 0$(或 $\beta f(a_1) \neq 0$), 那么 $a_1 f^{-1}(\beta) \neq 0$, 只可能 $a_1 f^{-1}(\beta) = ka_1$, $0 \neq k \in K$, $f(ka_1) = f(a_1)\beta \neq 0$, 与 $f(a_1)$ 是顶点的线性和矛盾.

引理 3.3 若有代数同构 $K(\Gamma) \simeq K(\Delta)$, 则有代数同构 $K(\Gamma'') \simeq K(\Delta'')$. $K(\Gamma''') \simeq K(\Delta''')$.

证明: 设 $f: K(\Gamma) \to K(\Delta)$ 是代数同构, 由引理3.2, 不妨设 Γ 和 Δ 中没有孤立点. 设 a 是 Γ'' 的顶点, $f(a)$ 的展开式中不能出现循环项. 设 $f(a) = \sum_i k_i b_i + \sum_i l_j \alpha_j$, $0 \neq k_i \in K$, $0 \neq l_j \in K$, α_j 是 Δ 中长度 $\geqslant 1$ 的非循环道路, 由 $f(a)^2 = f(a)$ 知, 存在 i, j, 使 $b_i \alpha_j \neq 0$(或者 $\alpha_j b_i \neq 0$), 那么 $f(a)\langle K(\Delta^*)^+\rangle \neq 0$, 那么 $a\langle K(\Gamma^*)^+\rangle \neq 0$, 矛盾. 说明 $f(a) = \sum_i k_i b_i$, b_i 是 Δ 的顶点, 同样道理可知 b_i 是 Δ'' 的顶点. 因而 $f\mid_{K(\Gamma'')}$: $K(\Gamma'') \to K(\Delta'')$ 是同构. 从而 $f: K(\Gamma)/K(\Gamma') \oplus K(\Gamma'') = K(\Gamma''') \to$

$K(\Delta)/K(\Delta')\oplus K(\Delta'')=K(\Delta''')$ 是同构.

引理 3.4 若有代数同构 $K(\Gamma')\simeq K(\Delta')$, 则 $\Gamma'\simeq\Delta'$.

证明是显然的. 从略.

设 $\Lambda=(\Lambda_0,\Lambda_1)$ 是有向图, 对 Λ_0 的子集 $\Lambda_0{}'$, $\Lambda(\Lambda_0{}')$ 表示 $\wedge$ 中 $\wedge{}'_0$ 以及与其相连的箭向所组成的子图.

引理 3.5 若有代数同构 $K(\Gamma'')\simeq K(\Delta'')$, 则 $\Gamma''\simeq\Delta''$.

证明: 设 $f:K(\Gamma'')\to K(\Delta'')$ 是代数同构. 设 a 是 Γ'' 的顶点, $f(a)=\sum_i k_ib_i+\sum_j l_jy_j$, $0\neq k_i\in K$, $0\neq l_j\in K$, b_2 是 Δ'' 的顶点, y_j 是 Δ'' 的长度 $\geqslant 1$ 的道路, 因而是孤立圈构成的循环. 由于 $f(a)^2=f(a)$, 易知 $f(a)=b_1+\cdots+b_n$, 因为 a 是本原幂等元, 所以 $n=1$, $f(a)=b$, b 是 Δ'' 的一个顶点. 那么 f 诱导了同构 $f:K(\Gamma''(a))\to K(\Delta''(b))$, 两个自由代数同构则它们的秩相等. 从而以 a 为顶点的孤立圈与以 b 为顶点的孤立圈等势.

引理 3.6 若 $f:K(\Gamma''')\to K(\Delta''')$ 是代数同构, 则

1) 设 a 是 $\Gamma*$ 的汇点, 则 $f(a)=b+\sum_{j\in J}l_j\beta_j$, 这里 b 是 Δ^* 的汇点, 对 $j\in J$, $l_j\in K$, β_j 是长度 $\geqslant 1$ 的非循环道路, β_j 的终点是 b.

2) 设 a 是 $\Gamma*$ 的源点, 则 $f(a)=b+\sum_{j\in J}l_j\beta_j$, 这里 b 是 Δ^* 的源点, 对 $j\in J$, $l_j\in K$, β_j 是长度 $\geqslant 1$ 的非循环道路, β_j 的终点是 b.

证明: 只证明 1), 设 a 是 Γ^* 的汇点, 可设

$$f(a)=\sum_{i\in I}k_ib_i+\sum_{j\in J}l_j\beta_j.$$

这里 $0\neq k_i\in K$, $0\neq l_j\in K$, b_i 是 Δ''' 的顶点, β_j 是 Δ''' 中长度 $\geqslant 1$ 的道路. 显然 $\{\beta_j\}_{j\in J}$ 中没有循环. 并且 $\{b_i\}_{i\in I}$ 是 Δ^* 的汇点集, 否则存在不是圈的箭向 β 使 $f(a)\beta\neq 0$, 那么 $af^{-1}(\beta)\neq 0$, 而 $f^{-1}(\beta)\in\langle K(\Gamma^*)^+\rangle$, 与 a 是 Γ^* 的汇点矛盾. 那么 $\left(\sum_i k_ib_i\right)\left(\sum_j l_j\beta_j\right)=0$, 因为 $f(a)^2=f(a)$, 所以

$\sum_{i\in I}k_i^2b_i+\left(\sum_{j\in J}l_j\beta_j\right)\left(\sum_{i\in I}k_ib_i\right)+\left(\sum_{j\in J}l_j\beta_j\right)^2\sum_{i\in I}k_ib_i+\sum_{j\in J}l_j\beta_j$, 特别地有 $\sum_{i\in I}k_i^2b_i=\sum_i k_ib_i$, 故 $k_i^2=k_i$, 那么 $k_i=1$, 所以 $f(a)=\sum_{i\in I}b_i+\sum_{j\in J}l_j\beta_j$.

令

$$J_0 = \{j \in J \mid \beta_j \text{不以任何} b_i \text{为终点}\}, J_i = \{j \in J \mid \beta_j \text{以} b_i \text{为终点}\}.$$

因为$\left(\sum_{j\in J} l_j\beta_j\right)\left(\sum_{i\in I} b_i\right) + \left(\sum_{j\in J} l_j\beta_j\right)^2 = \sum_{j\in J} l_j\beta_j$，那么$\left(\sum_{j\in J} l_j\beta_j\right)^2 = \sum_{j\in J_0} l_j\beta_j$，拆开就是

$$\left(\sum_{j\in J_0} l_j\beta_j\right)^2 + \left(\sum_{j\in J_0} l_j\beta_j\right)\left(\sum_{j\in J\setminus J_0} l_j\beta_j\right) + \left(\sum_{j\in J\setminus J_0} l_j\beta_j\right)\left(\sum_{j\in J_0} l_j\beta_j\right) + \left(\sum_{j\in J\setminus J_0} l_j\beta_j\right)^2$$
$$= \sum_{j\in J_0} l_j\beta_j$$

由此易知$\sum_{j\in J_0} l_j\beta_j = 0$，故$f(a) = \sum_{i\in I}\left(b_i + \sum_{j\in J_i} l_j\beta_j\right)$，$\left\{b_i + \sum_{j\in J_i} l_j\beta_j\right\}$是互相正交的幂等元，$a$是本原幂等元，故$f(a)$还是本原幂等元，故$f(a) = b + \sum_{j\in J} l_j\beta_j$，$b$是$\Delta^*$的汇点，$\beta_j$是长度$\geqslant 1$的以$b$为终点的非循环道路.

引理3.6中的b是由a所唯一确定的，我们记$b = \varphi_0(a)$.

设M_1是环R_1-模，M_2是环R_2-模，$g_1: R_1 \to R_2$是环同构，称加群同构$g: M_1 \to M_2$是g_1-同构，如果对任意$r_1 \in R_1, m_1 \in M_1$，有$g(r_1 m_1) = g_1(r_1)g(m_1)$.

设$f: K(\Gamma''') \to K(\Delta''')$是代数同构. Γ'''的顶点集等于Γ^*的顶点集. 对于Γ^*的两个不同的顶点a_1, a_2，从a_1到a_2的箭向张成的K-向量空间记为${}_{a_1}V_{a_2}$，简记为V，分以下几种情况：

1° a_1是Γ^*的源点，a_2是Γ^*的汇点，仿引理2.3易知，任意$A \in K[X]_{a_1} \bigotimes_K V \bigotimes_K K[X]_{a_2}$（$K[X]_{a_1}$表示以$a_1$为顶点的圈生成的自由代数），则$f(A) = B + C$，$B + C$的展开式中出现的道路起点是$\Delta^*$的源点，终点是$\Delta^*$的汇点，但是$f(A) = f(a_1)(B + C)f(a_2)$，由引理3.6知$B + C$是从$\varphi_0(a_1)$到$\varphi_0(a_2)$的道路的线性和. 记$\varphi_0(a_1)$到$\varphi_0(a_2)$箭向张成的向量空间为${}_{\varphi_0(a_1)}W_{\varphi_0(a_2)}$，我们记$B \in K[Y]_{\varphi_0(a_1)} \bigotimes_K W \bigotimes_K K[Y]_{\varphi_0(a_2)}$，而$C$的展开式中任意项不属于$K[Y]_{\varphi_0(a_1)} \bigotimes_K W \bigotimes_K K[Y]_{\varphi_0(a_2)}$，令$\varphi(A) = B$，若$A \neq 0$则$B \neq 0$.

2° a_1是Γ^*的源点，a_2不是Γ^*的汇点，$V \neq 0$，仿引理2.5知，存在Δ^*中唯一顶点，记为$\varphi_0(a_2)$，$\varphi_0(a_2)$不是Δ^*的汇点，对于任意$A \in K[X]_{a_1} \bigotimes_K V \bigotimes_K K[X]_{a_2}$，$f(A) = B + C + D$，$B + C$的展开式中出现的

道路起点是Δ^*的源点，终点是$\varphi_0(a_2)$，D的展开式中出现的道路起点是Δ^*的源点，终点是Δ^*的汇点，但是$f(A)=f(a_1)f(A)$，由引理3.6知$B+C+D$的展开式中出现的道路都以$\varphi_0(a_1)$为起点，我们记$B\in K[Y]_{\varphi_0(a_1)}\bigotimes_K W\bigotimes_K K[Y]_{\varphi_0(a_2)}$，而$C$的展开式中任意项不属于$K[Y]_{\varphi_0(a_1)}\bigotimes_K W\bigotimes_K K[Y]_{\varphi_0(a_2)}$，令$\varphi(A)=B$，若$A\neq 0$，则$B\neq 0$.

3° a_1不是Γ^*的源点，a_2是Γ^*的汇点，$V\neq 0$，存在Δ^*中唯一顶点，记为$\varphi_0(a_1)$，$\varphi_0(a_1)$不是Δ^*的源点，对于任意$A\in K[X]_{a_1}\bigotimes_K V\bigotimes_K K[X]_{a_2}$，$f(A)=B+C+D$，$B+C+D$的展开式中出现的道路都以$\varphi_0(a_2)$为终点，$B+C$的展开式中出现的道路起点是$\varphi_0(a_1)$，$D$的展开式中出现的道路起点是$\Delta^*$的源点；我们记$B\in K[Y]_{\varphi_0(a_1)}\bigotimes_K W_K K[Y]_{\varphi_0(a_2)}$，而$C$的展开式中任意项不属于$K[Y]_{\varphi_0(a_1)}\bigotimes_K W\bigotimes_K [Y]_{\varphi_0(a_2)}$，令$\varphi(A)=B$，若$A\neq 0$，则$B\neq 0$.

4° a_1不是Γ的源点，a_2不是Γ^*的汇点，$V\neq 0$，存在Δ^*中唯一确定的两个顶点，记为$\varphi_0(a_1)$，$\varphi_0(a_2)$，$\varphi_0(a_1)$不是Δ^*的源点，$\varphi_0(a_2)$不是Δ^*的汇点，对于任意$A\in K[X]_{a_1}\bigotimes_K V\bigotimes_K K[X]_{a_2}$，$f(A)=B+C+D+E+F$，$B+C$的展开式中出现的道路起点是$\varphi_0(a_1)$，终点是$\varphi_0(a_2)$，$D$的展开式中出现的道路起点是$\varphi_0(a_1)$，终点是$\Delta^*$的汇点，$E$的展开式中出现的道路起点是$\Delta^*$的源点，终点是$\varphi_0(a_2)$，$F$的展开式中出现的道路起点是$\Delta^*$的源点，终点是$\Delta^*$的汇点. 我们记$B\in K[Y]_{\varphi_0(a_1)}\bigotimes_K W\bigotimes_K K[Y]_{\varphi_0(a_2)}$，而$C$的展开式中任意项都不属于$K[Y]\text{-}_{\varphi_0(a_1)}\bigotimes_K W\bigotimes_K K[Y]_{\varphi_0(a_2)}$，令$\varphi(A)=B$，若$A\neq 0$，则$B\neq 0$.

当然要验证φ_0的合理性，这是平凡的工作. 实际上，对f^{-1}按上述方式导出的φ_0^{-1}和φ^{-1}满足：$\varphi_0\varphi_0^{-1}=1$，$\varphi_0^{-1}\varphi_0=1$，$\varphi\varphi^{-1}=1$，$\varphi^{-1}\varphi=1$.

我们还要建立K-代数同构$\varphi_1: K[X]_a\to K[Y]_{\varphi_0(a)}$.

i) 若a是Γ^*的汇点，任意$x\in K[X]_a$，因为$f(x)=f(x)f(a)$，由引理3.6知$f(x)$的展开式中出现的道路以$\varphi_0(a)$为终点，$f(x)=y+z$，y中出现的道路以$\varphi_0(a)$为起点，z中出现的道路不以$\varphi_0(a)$为起点，令$\varphi_1(x)=y$，显然φ_1是$K[X]_a$到$K[Y]_{\varphi_0(a)}$的代数同态，由f^{-1}得到

的 φ_1^{-1} 满足 $\varphi_1\varphi_1^{-1}=1,\varphi_1^{-1}\varphi_1=1$,所以 $\varphi_1:K[X]_a\to K[Y]_{\varphi_0(a)}$ 是代数同构.

ii) 若 a 是 Γ^* 的源点,类似于 i).

iii) 若 a 不是 Γ^* 的源点,也不是 Γ^* 的汇点,任意 $x\in K[Z]_a$,$f(x)=y+z_1+z_2+z_3$,$y\in K[Y]_{\varphi_0(a)}$,z_1 中出现的道路是以 $\varphi_0(a)$ 为顶点的循环,但不属于 $K[Y]_{\varphi_0(a)}$,z_2 中出现的道路不以 $\varphi_0(a)$ 为起点,z_3 中出现的道路不以 $\varphi_0(a)$ 为终点.若 z_2 中有一项是循环,不妨设有 Δ^* 中的箭向 β,β 的起点为 $\varphi_0(a')\neq\varphi_0(a)$,使 $f(x)\beta\neq 0$.那么 $xf^{-1}(\beta)\neq 0$,由上面的2°和4°知这是不可能的.所以 z_2 和 z_3 中都不能出现循环项,令 $\varphi_1(x)=y$,$\varphi_1:K[Z]_a\to K[Y]_{\varphi_0(a)}$ 是代数同态,由 f^{-1} 导出的 φ_1^{-1} 满足 $\varphi_1^{-1}\varphi_1=1,\varphi_1\varphi_1^{-1}=1$,故 $\varphi_1:K[Z]_a\to K[Y]_{\varphi_0(a)}$ 是代数同构.

容易知 $\varphi:K[Z]_{a_1}\bigotimes_K V\bigotimes_K K[Z]_{a_2}\to K[Y]_{\varphi_0(a_1)}\bigotimes_K W\bigotimes_K K[Y]_{\varphi_0(a_2)}$ 是双模 φ_1-同构.$K[X]_{a_1}\bigotimes_K \Lambda\bigotimes_K K[X]_{a_2}$ 作为 $K[Z]_{a_1}-K[Z]_{a_2}$ 一双模是自由模.

由[2]第114页知,$K[Z]_{a_1}\bigotimes_K K[Z]_{a_2}^{OP}$ 是所谓IBN环,因而$[K[X]_{a_1}\bigotimes_K V\bigotimes_K K[Z]_{a_2}:K[Z]_{a_1}{}^{n}_{K}K[Z]_{a_2}^{OP}]$ 有意义,显然$[V:K]=[K[Z]_{a_1}\bigotimes_K V\bigotimes_K K[X]_{a_2}^{OP}:K[Z]_{a_1}\bigotimes_K K[Z]_{a_2}^{OP}]$.

引理3.7 设 $f:K(\Gamma''')\to K(\Delta''')$ 是代数同构,则 $\Gamma'''\simeq\Delta'''$.

证明: 由以上讨论知,我们有 Γ''' 的顶点集到 Δ''' 的顶点集之间的一一对应 φ_0,满足对 Γ''' 的任意顶点a,$\varphi_1:K[Z]_a\simeq K[Y]_{\varphi_0(a)}$,那么这两个自由代数秩相等,即,以 a 为顶点的圈集合与以$\varphi_0(a)$为顶点的圈集合等势.对 Γ''' 的两个不同顶点 a_1,a_2,因为 $\varphi:K[Z]_{a_1}\bigotimes_K V\bigotimes_K K[Z]_{a_2}\to K[Y]_{\varphi_0(a_1)}\bigotimes_K W\bigotimes_K K[Y]_{\varphi_0(a_2)}$ 是双模 φ_1-同构,那么$[V:K]=[K[Z]_{a_1}\bigotimes_K V\bigotimes_K K[Z]_{a_2}:K[Z]_{a_1}\bigotimes_K K[Z]_{a_2}^{OP}]=[K[Y]_{\varphi_0(a_1)}\bigotimes_K W\bigotimes_K K[Y]_{\varphi_0(a_2)}:K[Y]_{\varphi_0(a_1)}\bigotimes_K K[Y]_{\varphi_0(a_2)}^{OP}]=[W:K]$,即从 a_1 到 a_2 的箭向集合与从 $\varphi_0(a_1)$ 到 $\varphi_0(a_2)$ 的箭向集合等势,故 Γ''' 与 Δ''' 同构.

总结引理3.2—3.5和引理3.7就有:

定理3.1 设 Γ 和 Δ 是任意两个有向图,K 是个域,若有 K-代数同构 $K(\Gamma)\simeq K(\Delta)$,则 Γ 和 Δ 同构.

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路代数的同构

Isomorphisms of Path Algebras

Liu Shaoxue, Luo Yunlun and Xiao Jie

Abstract Let $K(\Delta)$ denote the path algebra of a quiver Δ over a field K. It is proved that: (1) $K(\Delta)^+$ is isomorphic to $K(\Gamma)^+$ as K-algebras if and only if Δ and Γ are edge-isomorphic. (2) $K(\Delta) \simeq K(\Gamma)$ as K-algebras if and only if $\Delta \cong \Gamma$ as quivers.

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加法范畴的 Jacobson 结构定理*

The Jacobson Structure Theorem for Additive Categories

摘　要　本文从环论的角度来研究加法范畴的结构，对加法范畴完整地给出相应于环的 Jacobson 理论的结构定理. 定义加法范畴上的模与 Jacobson 根，证明本原加法范畴都是稠密线性变换加法范畴等.

§1. 有关加法范畴的一些概念

在[1]中我们曾以加法范畴为背景引入了 G－盟的概念，并定义了 G－盟的右理想及模等. 为了方便这里对加法范畴简述一下.

本文中范畴都是指加法范畴. 如无特别声明，它们的象元类都是 I(即其象元类与 I 之间有一个取定的一一对应). 范畴 A 的 $\mathrm{Hom}(\alpha,\beta)$，$\alpha,\beta\in I$，简记作${}_\alpha A_\beta$，常把范畴 A 写成 $A=\bigcup\limits_{\alpha,\beta}{}_\alpha A_\beta$.

设 $B=\bigcup\limits_{\alpha,\beta}{}_\alpha B_\beta\subseteq A$，其中${}_\alpha B_\beta\subseteq{}_\alpha A_\beta$. 若对 I 中任意α,β,γ，${}_\alpha B_\beta$ 都是${}_\alpha A_\beta$ 的子加群且${}_\alpha B_\beta\cdot{}_\beta A_\gamma\subseteq{}_\alpha B_\gamma$，则称 B 为A 的右理想. 类似可定义范畴 A 的左理想和理想.

设范畴 $A=\bigcup\limits_{\alpha,\beta\in I}{}_\alpha A_\beta$ 和$B=\bigcup\limits_{\alpha,\beta\in I}{}_\alpha B_\beta$. 特称 A 到B 的满足$\alpha\phi=\alpha$，$\forall\alpha\in I$ 的函子ϕ 为同态. 如果同态 ϕ 是一一对应，则称为 A 到B 的同构. 记 $\ker\phi$ 为A 中在同态ϕ 下映到O(表示任意${}_\alpha B_\beta$ 中的零元)的态射全体，易见 $\ker\phi$ 是A 的理想.

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设 $B = \bigcup_{\alpha,\beta} {}_\alpha B_\beta$ 是 $A = \bigcup_{\alpha,\beta} {}_\alpha A_\beta$ 的理想. 作 $\bar{A} = A/B = \bigcup_{\alpha,\beta} {}_\alpha \bar{A}_\beta$, 其中 ${}_\alpha\bar{A}_\beta = {}_\alpha A_\beta / {}_\alpha B_\beta$, 易见 $\bar{A}$ 是以 I 为象元类的一个加法范畴, 称之为 A 关于理想 B 的商范畴(参看[1,2]). 通常的同态定理在这里都是成立的.

把范畴 A 的右理想概念扩充到 A 的外部去便得 A 上右模的概念. 这就是, 设

$$M = \bigcup_{\alpha,\beta\in I} {}_\alpha M_\beta,$$

其中 ${}_\alpha M_\beta$ 是加群, 彼此不相交, 且 A 到 M 间有一个部分乘法运算: ma, $m \in {}_\gamma M_\delta$, $a \in {}_\alpha A_\beta$, 有定义当且仅当 $\delta = \alpha$, 此时 $ma \in {}_\gamma M_\beta$ (即 ${}_\gamma M_\alpha \cdot {}_\alpha A_\beta \subseteq {}_\gamma M_\beta$), 并在运算有定义时, 满足通常环模的条件, 即对任意 m, $n \in {}_\gamma M_\alpha$, $a, b \in {}_\alpha A_\beta$, $c \in {}_\beta A_\delta$, 有

$(m + n)a = ma + na, m(a + b) = ma + mb, m(bc) = (mb)c$, $m \cdot {}_\alpha 1_\alpha = m$,

其中 ${}_\alpha 1_\alpha$ 是环 ${}_\alpha A_\alpha$ 的单位元. 这时称 M 为加法范畴 A 的右模, 记作 A - 模 M. 易知 A 的右理想可自然地看成 A - 模.

M. Auslander[3] 曾把加法范畴 A 上的模定义为范畴 A 到全体 Abel 群的范畴 Ab 的函子, 即把模解释为范畴的表示, 和我们这里把模作为 A 的右理想的推广是不相同的. 当然这两种定义方式对于结合环是一致的.

设 M 是 A - 模. 若 $N = \bigcup_{\alpha,\beta} {}_\alpha N_\beta \subseteq M$, 且 ${}_\alpha N_\beta$ 是 ${}_\alpha M_\beta$ 的子加群而

$${}_\alpha N_\beta \cdot {}_\beta A_\gamma \subseteq {}_\alpha N_\gamma \ \forall\, \alpha, \beta, \gamma \in I,$$

则称 N 为 M 的子模. 易知子模 N 本身是 A - 模.

若 $M = \bigcup_\alpha {}_\beta M_\alpha$ (未写出的项都是加群 O, 以下同此), 其中 β 是固定元, 是右 A - 模, 则 M 给出 A 到 Ab 的一个函子 F: 对任意 $\alpha \in I$, $a \in {}_\delta A_\gamma$

$$\alpha F = {}_\beta M_\alpha$$

$$aF: {}_\beta M_\delta \to {}_\beta M_\gamma \quad x \mapsto xa.$$

反之, 任给 A 到 Ab 的一个函子 F, 任取定一个 $\beta \in I$, 令 $M = \bigcup_\alpha {}_\beta M_\alpha$, 其中 ${}_\beta M_\alpha = \alpha F$, 再适当定义 A 到 M 的部分乘法运算, 便得一个 A - 模 M. 这样, Auslander 意义下的 A - 模相当于我们这里的一类特殊形状的右 A - 模. (即左 A - 模 $M = \bigcup_\alpha {}_\alpha M_\beta$ 相当于 A 到 Ab 的反变函子, 而右 A - 模 $M = \bigcup_\alpha {}_\beta M_\alpha$ 相当于函子).

设 $M = \bigcup_{\alpha,\beta} {}_\alpha M_\beta$ 和 $N = \bigcup_{\alpha,\beta} {}_\alpha N_\beta$ 是 A - 模. 设 ϕ 是 M 到 N 的一个映射,

满足

1. $\phi_\alpha M_\beta \subseteq {}_\alpha N_\beta$, $\forall \alpha, \beta$. 若把 ϕ 在${}_\alpha M_\beta$ 上的限制记作${}_\alpha\phi_\beta$，它还是加群${}_\alpha M_\beta$ 到加群${}_\alpha N_\beta$ 的同态；

2. $\phi(ma) = \phi(m) \cdot a, \forall a \in A, m \in M$ 且 ma 有定义. 若 $m \in {}_\alpha M_\beta, a \in {}_\beta A_\gamma$，则 $ma \in {}_\alpha M_\gamma$，这就是${}_\alpha\phi_\gamma(ma) = {}_\alpha\phi_\beta(m) \cdot a$，则称 ϕ 为 A－模M 到N 的一个同态. 如果 ϕ 还是一一对应，则称为同构.

设 N 是A－模M 的子模. 作

$$\overline{M} = M/N = \bigcup_{\alpha,\beta} {}_\alpha\overline{M}_\beta,$$

其中${}_\alpha\overline{M}_\beta = {}_\alpha M_\beta / {}_\alpha N_\beta$. 对 $a \in {}_\alpha A_\beta, \overline{m} \in {}_\alpha\overline{M}_\beta, m \in {}_\alpha M_\beta$ 规定

$$\overline{m} \cdot a = \overline{m \cdot a}.$$

易见这样的规定是合理的且使 $\overline{M}$ 成为A－模. 称之为 M 关于子模N 的商模. 易知范畴 A 上的模的同态与商模有和环模完全类似的同态定理[1].

设 M 是A－模而$m \in M$. 规定

$$\mathrm{Ann} M = \{a \in A \mid \forall x \in M, \text{当 } xa \text{ 有定义时 } xa = 0\};$$

$$(0 : m) = \{a \in A \mid \text{或者 } ma \text{ 无定义，或者 } ma = 0\},$$

$$mA = \{ma \mid ma \text{ 有定义}\}.$$

直接验证知 Ann M 是A 的理想，$(0 : m)$ 是 A 的右理想，而 mA 是M 中含 m 的最小子模且有A－模同构

$$A/(0 : m) \simeq mA.$$

§2. 加法范畴的 Jacobson 根

$A = \bigcup_{\alpha,\beta} {}_\alpha A_\beta$ 永远表示以I 为象元类的加法范畴而$M = \bigcup_{\alpha,\beta} {}_\alpha M_\beta$ 永远表示 A－模. 当某些${}_\alpha M_\beta$ 为0 时，则常在并集表示中略去. A－模指右A－模.

定义 1　1) 若 A－模M 除0 和 M 外没有其他子模，则称之为单 A－模；

2) Ann $M = 0$ 的模 M 称为忠实模；

3) 称 $J(A) = \bigcap$ Ann M，其中 M 取遍所有单A－模，为加法范畴 A 的 Jacobson 根，易见 $J(A)$ 是 A 的理想；

4) $J(A) = 0$ 的 A 称作半本原加法范畴；

5) 具有忠实单 A－模的A 称作本原加法范畴.

若 $M = \bigcup_{\alpha,\beta} {}_\alpha M_\beta$ 是A-模，则任意取定β，${}_\beta M = \bigcup_\alpha {}_\beta M_\alpha$ 都是M的子模.同样，若

$$B = \bigcup_{\alpha,\beta} {}_\alpha B_\beta$$

是 A 的右理想，则

$${}_\beta B = \bigcup_\alpha {}_\beta B_\alpha$$

也是 A 的右理想.这样当 M 为单A-模时，必有 β 使

$$M = {}_\beta M = \bigcup_\alpha {}_\beta M_\alpha;$$

注意到 A 的右理想

$$B_i = \bigcup_{\alpha,\beta} {}_\alpha B^i_\beta,\quad i \in \sum(\sum \text{不必为集}),$$

之和
$$C = \sum_i B_i = \bigcup_{\alpha,\beta} {}_\alpha C_\beta,$$

其中 ${}_\alpha C_\beta = \sum_i {}_\alpha B^i_\beta({}_\alpha B^i_\beta, i \in \sum$，都在集${}_\alpha A_\beta$中，其中不同者的全体是一个集合)，

仍是右理想，而

$${}_\beta A = \bigcup_\alpha {}_\beta A_\alpha$$

都是 A 的右理想，易知 A 的极大右理想B 必具有形状：有一个 $\beta \in I$，

$$B = \bigcup_{\substack{\gamma\neq\beta\\ \alpha}} {}_\gamma A_\alpha \bigcup_\alpha {}_\beta B_\alpha.$$

若 $M = \bigcup_\alpha {}_\beta M_\alpha$ 是A-模，任取 $\gamma \in I$，作

$$N = \bigcup_\alpha {}_\gamma N_\alpha,$$

其中
$${}_\gamma N_\alpha = {}_\beta M_\alpha,\quad \forall\, \alpha,$$

仍用 A-模M 的运算，则有

$${}_\gamma N_\alpha \cdot {}_\alpha A_\delta = {}_\beta M_\alpha \cdot {}_\alpha A_\delta \subseteq {}_\beta M_\delta = {}_\gamma N_\delta,$$

由此易知 N 是A-模且 Ann M = Ann N.由于在讨论 A 的 Jacobson 根时，我们只对 Ann M 有兴趣，故常将单 A-模

$$M = \bigcup_\alpha {}_\beta M_\alpha$$

简写成
$$M = \bigcup_\alpha M_\alpha,$$

这时我们有
$$M_\alpha \cdot {}_\alpha A_\beta \subseteq M_\beta.$$

定理 1　$A \neq 0$ 是加法范畴，则有

(1) 单 A-模是存在的，因而 $J(A) \neq A$；

(2)$J(A)=\bigcap_i M_i$，其中 M_i 取遍 A 的一切极大右理想；

(3)$J(A)=\bigcup_{\alpha,\beta}{}_\alpha J_\beta$，则${}_\alpha J_\alpha = J({}_\alpha A_\alpha)$，$\forall\alpha$，其中 $J({}_\alpha A_\alpha)$ 是环${}_\alpha A_\alpha$ 的 Jacobson 根.

证明： (1) 由于 $A\neq 0$，必有 $\alpha\in I$，${}_\alpha A_\alpha\neq 0$. 因为环${}_\alpha A_\alpha$ 有单位元${}_\alpha\mathbf{1}_\alpha$，故它有极大右理想${}_\alpha R_\alpha$. 作

$$R'=\bigcup_{\substack{\gamma\neq\alpha\\ \beta}}{}_\gamma A_\beta\cup{}_\alpha R_\alpha\bigcup_{\beta\neq\alpha}{}_\alpha R_\alpha\cdot{}_\alpha A_\beta$$

直接验证知它是 A 的右理想. 取 A 中在(α,α) 处为${}_\alpha R_\alpha$ 的一切右理想而作它们的和

$$R=\bigcup_{\substack{\gamma\neq\alpha\\ \beta}}{}_\gamma A_\beta\cup{}_\alpha R_\alpha\bigcup_{\beta\neq\alpha}{}_\alpha R_\beta$$

易知 R 是右理想. 若有 A 的右理想 $R^*\supsetneqq R$，则 R^* 在(α,α) 处的加群${}_\alpha R^*_\alpha$ 必真包含${}_\alpha R_\alpha$，由于${}_\alpha R^*_\alpha$ 是环${}_\alpha A_\alpha$ 的右理想，由${}_\alpha R_\alpha$ 的极大性知${}_\alpha\mathbf{1}_\alpha\in{}_\alpha R^*_\alpha$，随之

$${}_\alpha\mathbf{1}_\alpha\cdot{}_\alpha A_\beta={}_\alpha A_\beta\subseteq R^*,\ \forall\beta,$$

因而 $R^*=A$. 这说明 R 是A 的一个极大右理想，A－模A 关于其极大子模 R 的商模即是一个单A－模. 故证得(1).

(2) 设 $M=\bigcup_\alpha M_\alpha$ 是单A－模. 任取 $m\neq 0\in M$，则 mA 是M 中含 m 的子模；由 M 的单性得$M=mA$. 另一方面 $mA\simeq A/(0:m)$. 总起来就证得$(0:m)$ 是 A 的极大右理想. 显然 $\mathrm{Ann}\,M=\bigcap_m(0:m)$，其中 m 取遍 M 中的非零元. 这样

$$J(A)=\bigcap_M\mathrm{Ann}\,M\supseteq\{A\text{ 的一切极大右理想之交}\}.$$

反之，设 R 是A 的一个极大右理想，则由前面讨论知，它具有下面形状：存在一个

$$\alpha\in I,$$
$$R=\bigcup_{\substack{\gamma\neq\alpha\\ \beta}}{}_\gamma A_\beta\cup{}_\alpha R_\alpha\bigcup_{\beta\neq\alpha}{}_\alpha R_\beta,$$

其中${}_\alpha R_\alpha\neq{}_\alpha A_\alpha$. 这样商模 A/R 是单A－模，并且

$$0\neq{}_\alpha\bar{\mathbf{1}}_\alpha\in{}_\alpha A_\alpha/{}_\alpha R_\alpha\subseteq A/R.$$

由之有 $$\mathrm{Ann}\,A/R\subseteq(0:{}_\alpha\bar{\mathbf{1}}_\alpha)=R,$$

最后一个等号是因为 $A/R={}_\alpha\bar{\mathbf{1}}_\alpha\cdot A$. 即有 $J(A)\subseteq R$. 即

$$J(A)\subseteq\{A\text{ 的一切极大右理想之交}\}.$$

与上面结果合起来便得(2).

(3) 设 $M=\bigcup_{\beta} M_{\beta}$ 是一个单 A - 模. 我们来考察 ${}_{\alpha}A_{\alpha}\cap \mathrm{Ann}M$. 为此只需考察 ${}_{\alpha}A_{\alpha}$ 作用到 M_{α} 的情况, 因为 ${}_{\alpha}A_{\alpha}$ 和 M 中其他 $M_{\beta}(\beta\neq\alpha)$ 之间没有运算. 当 $M_{\alpha}=0$ 时, 易知 ${}_{\alpha}A_{\alpha}\cap \mathrm{Ann}\ M={}_{\alpha}A_{\alpha}$; 当 $M_{\alpha}\neq 0$ 时则对任意 $m\neq 0\in M_{\alpha}$ 由 $mA=M$ 知 $m{}_{\alpha}A_{\alpha}=M_{\alpha}$, 即 M_{α} 是环 ${}_{\alpha}A_{\alpha}$ 的单模, 此时

$${}_{\alpha}A_{\alpha}\cap \mathrm{Ann}M=\text{单}{}_{\alpha}A_{\alpha}\text{-模}M_{\alpha}\text{的零化子}\supseteq J({}_{\alpha}A_{\alpha}).$$

即对任意单 A - 模 M 我们都有

$${}_{\alpha}A_{\alpha}\cap \mathrm{Ann}\ M\supseteq J({}_{\alpha}A_{\alpha}),$$

即有

$${}_{\alpha}A_{\alpha}\cap J(A)\supseteq J({}_{\alpha}A_{\alpha}).$$

反之, 若 N 是单右 ${}_{\alpha}A_{\alpha}$ - 模, 则有环 ${}_{\alpha}A_{\alpha}$ 的一个极大右理想 ${}_{\alpha}R_{\alpha}$ 使作为 ${}_{\alpha}A_{\alpha}$ - 模有

$${}_{\alpha}A_{\alpha}/{}_{\alpha}R_{\alpha}\simeq N.$$

由 ${}_{\alpha}R_{\alpha}$ 出发, 依(1) 中方法可得 A 的一个极大右理想

$$R=\bigcup_{\substack{\gamma\neq\alpha\\ \beta}}{}_{\gamma}A_{\beta}\cup{}_{\alpha}R_{\alpha}\bigcup_{\beta\neq\alpha}{}_{\alpha}R_{\beta},$$

这时单 A - 模

$$M=A/R=\bigcup_{\beta}{}_{\alpha}M_{\beta},$$

其中

$${}_{\alpha}M_{\alpha}\simeq{}_{\alpha}A_{\alpha}/{}_{\alpha}R_{\alpha}\simeq N(\text{作为}{}_{\alpha}A_{\alpha}\text{-模}).$$

由上面的讨论知

$$\begin{aligned}{}_{\alpha}A_{\alpha}\cap J(A)\subseteq{}_{\alpha}A_{\alpha}\cap \mathrm{Ann}M&=\text{单}{}_{\alpha}A_{\alpha}\text{-模}{}_{\alpha}M_{\alpha}\text{的零化子}\\&=\text{单}{}_{\alpha}A_{\alpha}\text{-模}N\text{的零化子}.\end{aligned}$$

随之有

$${}_{\alpha}A_{\alpha}\cap J(A)\subseteq J({}_{\alpha}A_{\alpha}).$$

总起来便是 ${}_{\alpha}A_{\alpha}\cap J(A)=J({}_{\alpha}A_{\alpha})$.

最后一节我们还要讨论 $J(A)$ 的性质.

定义 2　A 和 A_i, $i\in\Sigma$(Σ 不一定是集合) 都是以 I 为象元类的加法范畴, 称 $\{A_i, i\in\Sigma\}$ 为 A 的一个完全同态象类, 如果有同态 $\phi_i: A\to A_i$, 且

$$\bigcap_i \ker\phi_i=0.$$

由前面的讨论易证(我们略去证明).

定理 2　(1) 商范畴 $A/J(A)$ 是半本原加法范畴; (2) 半本原范畴有由本原范畴组成一个完全同态象类.

大家知道, M. Harada[2] 中对加法范畴 A 定义了 Jacobson 根, 是指 A

的具有性质

$$J \cap {}_\alpha A_\alpha = J({}_\alpha A_\alpha)$$

的任意理想 J. Harada 只是对于取有限直和闭的加法范畴讨论了 Jacobson 根的存在性和唯一性(参看[2]),对任意加法范畴的 Jacobson 根的存在性是未涉及的.因而本节以及最后一节关于任意加法范畴的 Jacobson 根的讨论是对 Harada 定义的补充和修饰.

§3.本原加法范畴的结构

若 ϕ 和 ψ 是 A-模 M 到自身的同态,则映射

$$\begin{rcases} \phi+\psi:(\phi+\psi)m = \phi m+\psi m, \\ \phi\psi:(\phi\psi)m = \phi(\psi m), \end{rcases} \forall\, m \in M$$

仍是 A-模 M 到自身的同态.其证法和环模情况一样.

命题 1 设 $M = \bigcup\limits_\alpha M_\alpha$ 是单 A-模,则

$$\mathrm{End}_A(M) = \{A\text{-模}M\text{ 的一切自同态}\}$$

关于同态的加法和乘法作成一个除环,且若 $M_\alpha \neq 0$,则

$$\mathrm{End}_A(M) \simeq \mathrm{End}_{{}_\alpha A_\alpha}(M_\alpha).$$

证明: 易见单 A-模 M 的任一非零自同态都是 M 的自同构.设对某个 α, $M_\alpha \neq 0$,由前知 M_α 是环 ${}_\alpha A_\alpha$ 的右单模.任意取定 $m_\alpha \neq 0 \in M_\alpha$,则由 A-模 M 的单性有

$$M = \bigcup_\beta m_\alpha \cdot {}_\alpha A_\beta.$$

易见 $\phi \in \mathrm{End}_A(M)$ 诱导出 ${}_\alpha A_\alpha$-模 M_α 的一个自同态 ψ,但 ϕ 由 $\phi m_\alpha \in M_\alpha$ 完全决定,故映射 $\phi \mapsto \psi$ 是 $\mathrm{End}_A(M)$ 到 $\mathrm{End}_{{}_\alpha A_\alpha}(M_\alpha)$ 的单射.这样可知道 $\mathrm{End}_A(M)$ 是一个集合.

另一方面,设 ψ 是单 ${}_\alpha A_\alpha$-模 M_α 的一个自同构.任取 $m_\alpha \neq 0 \in M_\alpha$,而令 $m_\alpha' = \psi m_\alpha$.由于 ψ 是 M_α 的 ${}_\alpha A_\alpha$-自同构,故 m_α 在环 ${}_\alpha A_\alpha$ 中的零化子,记作 $(0 : m_\alpha)_{{}_\alpha A_\alpha}$,必等于 m_α' 的,即有

$$(0 : m_\alpha)_{{}_\alpha A_\alpha} = (0 : m_\alpha')_{{}_\alpha A_\alpha} = {}_\alpha B_\alpha,$$

其中 ${}_\alpha B_\alpha$ 是环 ${}_\alpha A_\alpha$ 的一个极大右理想.今证 m_α 和 m_α' 在 A 中的零化子也相等,即

$$(0 : m_\alpha) = (0 : m_\alpha').$$

首先它们都是 A 的极大右理想且在(α,α)处都是${}_\alpha B_\alpha$,即

$$(0:m_\alpha)=\bigcup_{\substack{\gamma\neq\alpha\\ \delta}}{}_\gamma A_\delta\cup{}_\alpha B_\alpha\bigcup_{\beta\neq\alpha}{}_\alpha B_\beta,$$

$$(0:m_\alpha{}')=\bigcup_{\substack{\gamma\neq\alpha\\ \delta}}{}_\gamma A_\delta\cup{}_\alpha B_\alpha\bigcup_{\beta\neq\alpha}{}_\alpha B_\beta{}'.$$

作此二右理想之和 C,即

$$C=\bigcup_{\substack{\gamma\neq\alpha\\ \delta}}{}_\gamma A_\delta\cup{}_\alpha B_\alpha\bigcup_{\beta\neq\alpha}({}_\alpha B_\beta+{}_\alpha B_\beta{}'),$$

则 $C\neq A$ 且$(0:m_\alpha)\subseteq C$及$(0:m_\alpha{}')\subseteq C$,因而 $C=(0:m_\alpha)=(0:m_\alpha{}')$.注意到 $M=m_\alpha A$,依下法 ψ 扩充为A-模M的一个对应ϕ:$\phi(m_\alpha\cdot{}_\alpha a_\beta)=m_\alpha{}'\cdot{}_\alpha a_\beta$, $\forall\,{}_\alpha a_\beta\in A$.这个对应定义是合理的,因为由$(0:m_\alpha)=(0:m_\alpha{}')$,故有 $m_\alpha\cdot{}_\alpha a_\beta=0$.当且仅当 $m_\alpha{}'\cdot{}_\alpha a_\beta=0$.易见 ϕ 是A-模M的自同构.显然 ϕ 在M_α上诱导的M_α的${}_\alpha A_\alpha$-自同构就是给定的ψ.这样对应 $\phi\to\psi$ 是满的,而给出 $\mathrm{End}_A(M)$ 和 $\mathrm{End}_{{}_\alpha A_\alpha}(M_\alpha)$ 之间的同构对应.

定义 1 以除环 D 上所有左向量空间M_α, $\alpha\in\Sigma$,为象元,以所有这些空间之间的所有 D-线性变换为态射的加法范畴C称作D上全线性变换加法范畴.C 的一个加法子范畴 $A=\bigcup_{\alpha,\beta\in I}{}_\alpha A_\beta$,其中 $I\subseteq\Sigma$ 而${}_\alpha A_\beta\subseteq\mathrm{Hom}_D(M_\alpha,M_\beta)$,叫作稠密线性变换范畴,如果对所有 $\alpha,\beta\in I$, ${}_\alpha A_\beta$ 在 $\mathrm{Hom}_D(M_\alpha,M_\beta)$ 中是稠密的,即指:对任意自然数 n,以及任意从 M_α 中取定的 D-无关元$x_1,x_2,\cdots,x_n$,和任意从 M_β 中取定的元素$y_1,y_2,\cdots,y_n$,必有 $a\in{}_\alpha A_\beta$,使 $x_ia=y_i$, $i=1,2,\cdots,n$.

设 A 是以I为象元类的加法范畴, $F=\{\alpha_1,\cdots,\alpha_n\}$ 是 I 的一个有限子集.现在我们来构造一个新加法范畴 B,其象元类为 $I_1=I\setminus F$ 以及一个不在 I 中的新符号 π,其 Hom 集${}_\alpha B_\beta$ 规定如下:

$${}_\alpha B_\beta={}_\alpha A_\beta\quad 当\ \alpha,\beta\in I_1\ 时,$$

$${}_\beta B_\pi=\bigoplus_{\alpha\in F}{}_\beta A_\alpha\quad 当\ \beta\in I_1\ 时,$$

$${}_\pi B_\beta=\bigoplus_{\alpha\in F}{}_\alpha A_\beta\quad 当\ \beta\in I_1\ 时,$$

$${}_\pi B_\pi=\bigoplus_{\alpha,\beta\in F}{}_\alpha A_\beta$$

利用 A 中态射的加法和乘法,直接验证知 B 是一个加法范畴.直观地说 B 由A 把其中象元$\alpha_1,\alpha_2,\cdots,\alpha_n$ 合并成一个新象元 π 而得到的.

下面我们来考察 A-模和B-模之间的关系.设 $M=\bigcup_{\alpha\in I}M_\alpha$ 是单A-模.作

$$N = \bigcup_{\alpha \in I_1} N_\alpha \cup N_\pi,$$

其中 $$N_\alpha = M_\alpha \quad \forall \alpha \in I_1, N_\pi = \bigoplus_{\alpha \in F} M_\alpha.$$

利用A-模M的运算,可使N成为一个B-模,即当用$\circ$表示B-模N的运算,用$\cdot$表示A-模M的运算,我们规定:当$x \in N_\alpha, b \in {}_\alpha B_\beta, \alpha, \beta \in I_1$时$x \circ b = x \cdot b$;当

$$x = \bigoplus_{\alpha \in F} x_\alpha \in N_\pi,$$

其中$x_\alpha \in N_\alpha$,

$$b = \bigoplus_{\alpha \in F} {}_\alpha a_\beta \in {}_\pi B_\beta,$$

其中${}_\alpha a_\beta \in {}_\alpha A_\beta$时,

$$x \circ b = \bigoplus_{\alpha \in F} x_\alpha \cdot {}_\alpha a_\beta;$$

当$x = \bigoplus_{\alpha \in F} x_\alpha \in N_\pi$,其中$x_\alpha \in N_\alpha$,

$$b = \bigoplus_{\alpha, \beta \in F} {}_\alpha a_\beta,$$

其中${}_\alpha a_\beta \in {}_\alpha A_\beta$时

$$x \circ b = \bigoplus_{\alpha \in F} \left(\sum_{\beta \in F} x_\beta \cdot {}_\beta a_\alpha \right).$$

直接验算可知N是一个B-模.

引理1 A, B, M, N之意义如上,则有

(1) 若M是单A-模,则N是单B-模;

(2) 若$\operatorname{Ann} M = \bigcup_{\alpha, \beta \in I} {}_\alpha A_\beta'$,而

$$\operatorname{Ann} N = \bigcup_{\alpha, \beta \in I_1} {}_\alpha B_\beta' \bigcup_{\alpha \in I_1} {}_\alpha B_\pi' \bigcup_{\alpha \in I_1} {}_\pi B_\alpha' \cup {}_\pi B_\pi',$$

则 $${}_\alpha A_\beta' = {}_\alpha B_\beta', \forall \alpha, \beta \in I_1; \bigoplus_{x \in F} {}_\alpha A_x' = {}_\alpha B_\pi';$$

$$\bigoplus_{x \in F} {}_x A_\alpha' = {}_\pi B_\alpha'; \bigoplus_{x, y \in F} {}_x A_y' = {}_\pi B_\pi'.$$

证明:(1) 欲证N是单B-模,只需证对N中任意$x \neq 0$, $xB = N$.若$x \in N_\beta = M_\beta, \beta \in I_1$,此时显然有$xB \supseteq M_\alpha, \alpha \in I$,故$xB = N$.若$x \in N_\pi$,则

$$x = \bigoplus_\alpha x_\alpha,$$

其中 $$x_\alpha \in N_\alpha, \alpha \in F.$$

由$x \neq 0$,知有一$x_\beta \neq 0$.取${}_\pi B_\pi$中的元素${}_\beta \mathbf{1}_\beta$(环${}_\beta A_\beta$的单位元),则有

$$x \circ {}_\beta \mathbf{1}_\beta = x_\beta \in N,$$

随之有 $$xB \supseteq x_\beta B = N.$$

即 N 是单 B - 模.

(2) 直接验证可得.

反之,设 $N=\bigcup_{\beta\in I_1}N_\beta\cup N_\pi$ 是 B - 模. 由于 ${}_\pi B_\pi$ 的单位元

$$ {}_\pi\mathbf{1}_\pi=\bigoplus_{\alpha\in F}{}_\alpha\mathbf{1}_\alpha $$

而 ${}_\alpha\mathbf{1}_\alpha$, $\alpha\in F$, 是正交幂等元集, 故有

$$ N_\pi=N_\pi\circ{}_\pi\mathbf{1}_\pi=\bigoplus_{\alpha\in F}N_\alpha, $$

其中

$$ N_\alpha={}_\pi N_\pi\circ{}_\alpha\mathbf{1}_\alpha. $$

对 $I=I_1\cup\{\alpha_1,\cdots,\alpha_n\}$ 中任意 α, 令 $M_\alpha=N_\alpha$ 而作

$$ M=\bigcup_{\alpha\in I}M_\alpha. $$

利用 B - 模 N 的运算, 与上类似地可将 M 作成一个 A - 模, 类似地有

引理 2　A,B,N,M 的意义如上. 若 N 是单 B - 模, 则 M 是单 A - 模. 且若

$$ \operatorname{Ann}N=\bigcup_{\alpha,\beta\in I_1}{}_\alpha B_\beta{}'\bigcup_{\alpha\in I_1}{}_\alpha B_\pi{}'\bigcup_{\alpha\in I_1}{}_\pi B_\alpha{}'\cup{}_\pi B_\pi{}', $$

$$ \operatorname{Ann}M=\bigcup_{\alpha,\beta\in I}{}_\alpha A_\beta{}', $$

则

$$ {}_\alpha B_\beta{}'={}_\alpha A_\beta{}',\quad\forall\alpha,\beta\in I_1; $$

$$ {}_\alpha B_\pi{}'\cdot{}_x\mathbf{1}_x={}_\alpha A_x{}',\quad\forall\alpha\in I_1,x\in F; $$

$$ {}_x\mathbf{1}_x\cdot{}_\pi B_\alpha{}'={}_x A_\alpha{}',\quad\forall x\in F,x\in I_1; $$

$$ {}_x\mathbf{1}_x\cdot{}_\pi B_\pi{}'\cdot{}_y\mathbf{1}_y={}_x A_y{}',\quad\forall x,y\in F. $$

引理 3　A,B,M,N 的意义如上. 此时有

$$ \operatorname{End}_A(M)\simeq\operatorname{End}_B(N). $$

证明: $\operatorname{End}_A(M)$ 中的 ϕ 可诱导出 B - 模 N 的一个自同态 ϕ', 规定:

$$ \phi'x=\phi x,\quad x\in N_\alpha,\alpha\in I_1; $$

对于

$$ x=\bigoplus_\alpha x_\alpha\in N_\pi, $$

其中

$$ x_\alpha\in M_\alpha,\alpha\in F, $$

规定

$$ \phi'x=\bigoplus_\alpha\phi x_\alpha. $$

反之, 若 ϕ' 是 B - 模 N 的一个自同态, 则由 $\phi'N_\alpha\subseteq N_\alpha$, $\alpha\in I_1$, 以及

$$ \phi'N_\pi=\phi'(\bigoplus_{\alpha\in F}M_\alpha)\subseteq N_\pi=\bigoplus_{\alpha\in F}M_\alpha $$

因而对 $\alpha\in F=\{\alpha_1,\cdots,\alpha_n\}$ 也有

$$ \phi'M_\alpha=\phi'(N_\pi\circ{}_\alpha\mathbf{1}_\alpha)\subseteq(\phi'N_\pi)\circ{}_\alpha\mathbf{1}_\alpha\subseteq N_\pi\circ{}_\alpha\mathbf{1}_\alpha=M_\alpha, $$

故只要规定 $\phi x = \phi' x, \quad \forall x \in M_\alpha, \forall \alpha \in I,$
便得 A - 模 M 的一个自同态，因为 A 对 M 的作用和 B 对 N 的作用是一样的，故 ϕ 保持 A - 模 M 中的运算是不成问题的. 对应 $\phi \leftrightarrow \phi'$ 给出了 $\mathrm{End}_A(M)$ 和 $\mathrm{End}_B(N)$ 的同构.

作为引理 1,2 和上节定理 1 的推论，有

定理 1 A, B 的意义如上. 若

$$J(A) = \bigcup_{\alpha,\beta \in I} {}_\alpha A_\beta{}',$$

$$J(B) = \bigcup_{\alpha,\beta \in I_1} {}_\alpha B_\beta{}' \bigcup_{\alpha \in I_1} {}_\alpha B_\pi{}' \bigcup_{\alpha \in I_1} {}_\pi B_\alpha{}' \bigcup {}_\pi B_\pi{}',$$

则有

$$ {}_\alpha B_\beta{}' = {}_\alpha A_\beta{}' \quad \alpha, \beta \in I_1,$$

$$ {}_\alpha B_\pi{}' = \bigoplus_{\beta \in F} {}_\alpha A_\beta{}', \quad \alpha \in I_1,$$

$$ {}_\pi B_\alpha{}' = \bigoplus_{\beta \in F} {}_\beta A_\alpha{}', \quad \alpha \in I_1,$$

$$ {}_\pi B_\pi{}' = \bigoplus_{\alpha,\beta \in F} {}_\alpha A_\beta{}';$$

并且

$$J({}_\pi B_\pi) = J(B) \cap {}_\pi B_\pi = {}_\pi B_\pi{}' = \bigoplus_{\alpha,\beta \in F} {}_\alpha A_\beta{}'.$$

下面是本原加法范畴的主要结构定理.

定理 2 (1) 本原加法范畴必同构于一个稠密线性变换加法范畴；

(2) 稠密线性变换加法范畴都是本原加法范畴.

证明:(2) 是容易证明的，下面只给出(1) 的证明.

设 $A = \bigcup_{\alpha,\beta \in I} {}_\alpha A_\beta$ 有一个忠实单 A - 模

$$M = \bigcup_{\alpha \in I} M_\alpha.$$

令

$$D = \mathrm{End}_A(M).$$

由命题 1 知 D 是除环. 由 $DM_\alpha \subseteq M_\alpha$，故知 M_α 可看作 D - 左向量空间，而 ${}_\alpha A_\beta$ 中的元素可解释成为 D - 空间 M_α 到 M_β 的 D - 线性变换. 由于 ${}_\alpha A_\beta$ 中元素只与 M_α 有运算，故由 A - 模 M 的忠实性知，当 $a \in {}_\alpha A_\beta$, $M_\alpha a = 0$ 当且仅当 $a = 0$，即 ${}_\alpha A_\beta$ 可看成 $\mathrm{Hom}_D(M_\alpha, M_\beta)$ 的子集.

首先证明 ${}_\alpha A_\alpha$ 在 $\mathrm{Hom}_D(M_\alpha, M_\alpha)$ 中是稠密的. 如果 $M_\alpha = 0$，则没有什么要证的. 若 $M_\alpha \neq 0$，则由 A - 模 M 的单性知对任意 $0 \neq m \in M_\alpha$ 有 $m {}_\alpha A_\alpha = M_\alpha$，故 ${}_\alpha A_\alpha \neq 0$ 且 M_α 是环 ${}_\alpha A_\alpha$ 的忠实单模. 设 $D_\alpha = \mathrm{End}_{{}_\alpha A_\alpha}(M_\alpha)$. 依命题 1 在对应 $\phi \to \psi$ 下有 $D \simeq D_\alpha$，且 ψ 就是 ϕ 在 M_α 上的限制，故若按此同构对应把 D 和 D_α 等同起来，则它们对 M_α 的作用是一致的. 这样 M_α 中一元素集是 D - 无关和是 D_α - 无关是完全一样的. 由 Chevalley-Jacobson

稠密定理知${}_\alpha A_\alpha$ 在D_α-空间M_α 上的作用是稠密的,故知${}_\alpha A_\alpha$ 在D-空间M_α 上的作用是稠密的,即对 M_α 中任意有限D-无关(因而 D_α-无关)集 $x_1,\cdots,x_n$ 及M_α 中任意$y_1,\cdots,y_n$ 必有$a\in{}_\alpha A_\alpha$ 使$x_ia=y_i$, $\forall i$,即${}_\alpha A_\alpha$ 在 $\mathrm{Hom}_D(M_\alpha,M_\alpha)$ 中稠密.

其次证明${}_\alpha A_\beta$ 在$\mathrm{Hom}_D(M_\alpha,M_\beta)$ 中也是稠密的.为此取$\alpha_1=\alpha,\alpha_2=\beta$ 如前由A 作加法范畴B,这里

$$B=\bigcup_{\gamma,\delta\in I_1}{}_\gamma B_\delta\bigcup_{\gamma\in I_1}{}_\gamma B_\pi\bigcup_{\gamma\in I_1}{}_\pi B_\gamma\cup{}_\pi B_\pi,$$

$${}_\pi B_\pi={}_\alpha A_\alpha\oplus{}_\alpha A_\beta\oplus{}_\beta A_\alpha\oplus{}_\beta A_\beta,$$

由 M 作N,这里

$$N=\bigcup_{\gamma\in I_1}N_\gamma\cup N_\pi,$$

$$N_\pi=M_\alpha\oplus M_\beta.$$

由引理 1 知 N 是忠实单B-模,由引理 3 知

$$D=\mathrm{End}_A(M)=\mathrm{End}_B(N)$$

(这里把它们按该同构等同起来).

把上面证明结果应用于 B 中的${}_\pi B_\pi$,便得${}_\pi B_\pi$ 在$\mathrm{Hom}_D(N_\pi,N_\pi)$ 内是稠密的.因而在$M_\alpha\subseteq N_\pi$ 中任取有限D-无关集$x_1,\cdots,x_n$ 及$M_\beta\subseteq N_\pi$ 中任意元 $y_1,\cdots,y_n$ 必有

$$b={}_\alpha b_\alpha+{}_\alpha b_\beta+{}_\beta b_\alpha+{}_\beta b_\beta\in{}_\pi B_\pi,$$

其中${}_\gamma b_\delta\in{}_\gamma A_\delta$,使得

$$x_ib=y_i,\quad\forall i.$$

注意到 $M_\delta\cdot{}_\delta A_\gamma\subseteq M_\gamma$,由之便得

$$x_\alpha b_\beta=y_i,\quad\forall i.$$

即证得${}_\alpha A_\beta$ 在 $\mathrm{Hom}_D(M_\alpha,M_\beta)$ 中的稠密性.

定义 2 称 A 为右 Artin 加法范畴,如果 A 中任意右理想降链

$$R_1\supseteq R_2\supseteq\cdots\supseteq R_n\supseteq\cdots$$

在有限处中断.单加法范畴指无真正理想的范畴.

作为上面定理的推论,有

定理 3 单右 Artin 加法范畴 A 必同构于某一除环D 上有限维左向量空间加法范畴的一个完全子范畴且其象元集是有限的.

证明: 首先证明 Artin 范畴 A 的象元集I 是有限的.否则取不同的

$$\alpha_i \in I, \quad i = 1,2,\cdots$$

令
$$R_n = \bigcup_{i \geqslant n} {}_{\alpha_i}A_\beta,$$

则 $R_n, n = 1,2,\cdots$ 组成无限降链，这与 A 是 Artin 的相矛盾.

由于 A 只有有限个象元，又对右理想满足降链条件，故 A 有极小右理想R. 由于 A 是单的，知 A－模R 是忠实单模，因而 A 是稠密线性变换范畴. 若 A 中有一个象元M_α 是D 上无限维向量空间，则取 M_α 中可数个 D- 无关元$x_i, i = 1,2,\cdots$，而令

$$ {}_\alpha R_\alpha^n = \{ a \in {}_\alpha A_\alpha \mid x_i a = 0, i = 1,2,\cdots,n \}.$$

易见${}_\alpha R_\alpha^n \cdot {}_\alpha A_\alpha \subseteq {}_\alpha R_\alpha^n$. 作

$$R_n = \bigcup_\beta {}_\alpha Q_\beta,$$

而令${}_\alpha Q_\beta = {}_\alpha R_\alpha^n \cdot {}_\alpha A_\beta, \forall \beta$. 直接验证知 R_n 是A 的右理想且在(α,α) 处为${}_\alpha R_\alpha^n$. 由${}_\alpha A_\alpha$ 在 $\mathrm{Hom}_D(M_\alpha, M_\alpha)$ 中的稠密性知${}_\alpha R_\alpha^{n+1} \subsetneq {}_\alpha R_\alpha^n, \forall_n$，故得 A 中无限递降右理想

$$R_1 \supsetneq R_2 \supsetneq \cdots \supsetneq R_n \supsetneq \cdots$$

这与是右 Artin 相矛盾，即知 A 的每一象元都是有限维的.

§4. 加法范畴的 Jacobson 根(续)

在 §2 中我们是用右 A－模来定义A 的 Jacobson 根，而有

$$J(A) = \bigcap \mathrm{Ann} M, M \text{ 取遍右单} A－\text{模},$$

$$J({}_\pi A_\pi) = {}_\pi A_\pi \cap J(A) (\text{参看 §3 定理 1}), \qquad (*)$$

其中$J({}_\pi A_\pi)$ 表示环${}_\pi A_\pi$ 的右Jacobson 根(即用环的右模定义者). 同样，也可以用左 A－模来定义A 的 Jacobson 根，称之为 A 的左Jacobson 根，记作 $J'(A)$. 类似地去讨论，我们也有

$$J'(A) = \bigcap \mathrm{Ann} M, M \text{ 取遍左单} A－\text{模},$$

$$J'({}_\pi A_\pi) = {}_\pi A_\pi \cap J'(A), \qquad (*)$$

其中 $J'({}_\pi A_\pi)$ 表示环${}_\pi A_\pi$ 的左 Jacobson 根.

但对环${}_\pi A_\pi$ 言，其左，右 Jacobson 根是相等的，因而如果取 $\gamma = \alpha_1$，$\delta = \alpha_2$(参看 §3)，

此时
$${}_\pi A_\pi = {}_\gamma A_\gamma \oplus {}_\gamma A_\delta \oplus {}_\delta A_\gamma \oplus {}_\delta A_\delta,$$

而令
$$J(A) = \bigcup_{\alpha,\beta} {}_\alpha J_\beta, J'(A) = \bigcup_{\alpha,\beta} {}_\alpha J_\beta{}',$$

则由(*)有 $_{\chi}J_{\gamma} \oplus _{\chi}J_{\delta} \oplus _{\delta}J_{\gamma} \oplus _{\delta}J_{\delta}, = J(_{\pi}A_{\pi}) = J'(_{\pi}A_{\pi}) = _{\chi}J_{\gamma}' \oplus _{\chi}J_{\delta}' \oplus _{\delta}J_{\gamma}' \oplus _{\delta}J_{\delta}'$,

故有 $_{\chi}J_{\delta} = _{\chi}J_{\delta}', _{\chi}J_{\gamma} = _{\chi}J_{\gamma}'$，即 $J(A) = J'(A)$. 故有

定理 1 加法范畴的右 Jacobson 根和左 Jacobson 根是相等的.

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有向图的几何性质和其路代数的代数性质*①

Geometric Properties of Directed Graphs and Algebraic Properties of Their Path Algebras

摘　要　本文研究有向图Δ的几何性质和其路代数 $K(\Delta)$ 的代数性质之间的关系. 给出有向图Δ的路代数 $K(\Delta)$ 是 Artin 代数, Noether 代数, 半本原代数以及素代数的充分必要条件.

一

我们在[1] 中证明了两个有向图是同构的当且仅当它们的路代数(在同一域 K 上的) 是同构的. 这样, 有向图的几何性质(即在有向图的自同构下不变的性质) 和其路代数的代数性质(即在路代数的自同构下不变的性质) 是彼此对应相互决定的. 本文的目的在于给出这方面的一些结果, 说明路代数该是研究有向图的一个有用工具.

首先引叙一下我们下面要用的概念.

一个有向图是 $\Delta = (\Delta_0, \Delta_1, s, e)$, 其中 Δ_0 和 Δ_1 是两个集合(有限或无限), s, e 是从 Δ_j 到 Δ_0 的映射. 称 Δ_0 是Δ的顶点集, 其中元素用 $a, b, \cdots$ 表示, Δ_1 是Δ的箭向集, 其中元素用 $\alpha, \beta, \cdots$ 表示. 对 $\alpha \in \Delta_1$, $s(\alpha)$, $e(\alpha)$ 分别叫作 α 的起点和终点.

称 $(a \mid \alpha_1, \cdots, \alpha_l \mid b)$, 其中 a 是箭向 α_1 的起点, b 是 α_l 的终点, 且对 $1 \leqslant i \leqslant l-1$ 有 $e(\alpha_i) = s(\alpha_{i+1})$, 为有向图Δ中从 a 到 b 的长度 $= l$ 的道路(或路). 这样箭向 α 是长度为1的道路 $(a \mid \alpha \mid b)$. 我们还引入长度为

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零的道路$(a \mid a)$, $a \in \Delta_0$. 一个从a到a的长度$\geqslant 1$的道路称为循环道路.

取定域K, 有向图Δ的路代数$K(\Delta)$是指: 首先$K(\Delta)$是由Δ中所有道路为基作成的K-向量空间, 乘法由道路之间的乘法按分配律给出, 两个道路的乘法规定如下:

$$(a \mid \alpha_1, \cdots, \alpha_l \mid b) \cdot (c \mid \beta_1, \cdots, \beta_m \mid d)$$
$$= \begin{cases} (a \mid \alpha_1, \cdots, \alpha_l, \beta_1, \cdots, \beta_m \mid d), \text{当 } b = c \\ 0, \text{当 } b \neq c. \end{cases}$$

按这种方式我们得到K上结合代数$K(\Delta)$.

道路$u = (a \mid \alpha_1, \cdots, \alpha_l \mid b)$的子道路是指形如$v = (a' \mid \alpha_i, \cdots, \alpha_m \mid b')$, $i \geqslant 1$, $m \leqslant l$的路. 此时也说u是v的扩道路. 易见, u的子道路v是u表成一些道路乘积时的一个因子.

任取$x \in K(\Delta)$, 则x是Δ中一些不同道路的K-线性组合. 称此表达式中具有非零系数的道路的长度中的极大者为元素x的长度, 易知, 若$x, y \in K(\Delta)$且$x(y)$中出现的道路都以a为终点(起点)时,

$$x \cdot y \text{ 的长度} = x \text{ 的长度} + y \text{ 的长度}.$$

用$K(\Delta)^+$表示Δ_1在$K(\Delta)$中生成的子代数.

二

很容易证明下面命题.

命题 1 下列命题等价

1) 有向图Δ没有循环道路;

2) 路代数$K(\Delta)$是局部有限维的;

3) $K(\Delta)^+$是诣零的.

证明: 1)⇒2). 只需证明由有限个箭向$\alpha_1, \cdots, \alpha_n$在$K(\Delta)$中所生成的子代数$A$是有限维的就够了. 由于$\Delta$中没有循环道路, 故由$\alpha_1, \cdots, \alpha_n$中若干个(可重复使用)所组成的非零道路, 其长度不能超过n, 否则将有某一α_i在该道路中至少出现两次, 随之它含有一个子道路是循环道路. 这样子代数A的维数$\leqslant n^n$, 故得.

1)⇒3). 显然任意非循环道路的平方都为零. 设$x \in K(\Delta)^+$是任意元素, 则它是一些非循环道路的线性和, 若在这些道路中出现的不同箭向共有n个, 与上类似地可知: $x^{n+1} = 0$.

2)⇒1),3)⇒1) 是显然的.

命题 2 下列命题等价

1) 有向图 Δ 只有有限个顶点,有限个箭向且无循环道路;

2) $K(\Delta)$ 是有限维代数;

3) $K(\Delta)$ 是左(或右)Artin 代数.

证明:1),2) 等价由命题 1 即得. 剩下来要证的只是 3)⇒1). 用 $(S\rangle$ 表示 S 在 $K(\Delta)$ 中生成的左理想.

若 Δ 有无限个顶点,令 $e_i = \{a_i \mid a_i\}, i = 1,2,\cdots, S_n = \{e_i, i \geqslant n\}$,则有无限递降左理想链,

$$(S_1\rangle \supsetneqq (S_2\rangle \supsetneqq \cdots \supsetneqq (S_n\rangle \supsetneqq \cdots$$

若 Δ 有无限个不同箭向 $\alpha_i, i = 1,2,\cdots$,令 $T_n = \{\alpha_i, i \geqslant n\}$,便得无限递降左理想链

$$(T_1\rangle \supsetneqq (T_2\rangle \supsetneqq \cdots \supsetneqq (T_n\rangle \supsetneqq \cdots$$

若 Δ 有循环道路 $x = \{a \mid \alpha_1, \cdots, \alpha_n \mid a\}$,则得无限递降左理想链

$$(x\rangle \supsetneqq (x^2\rangle \supsetneqq (x^3\rangle \supsetneqq \cdots$$

即证得 3)⇒1).

命题 3 下列命题等价

1) 有向图 Δ, i) 只有有限个顶点, 有限个箭向; ii) 对任意循环道路 $(a \mid \alpha_1, \cdots, \alpha_n \mid a)$ 不存在箭向 $\alpha = (a \mid \alpha \mid b) \neq \alpha_1$;

2) $K(\Delta)$ 是右 Noether 代数.

证明:2)⇒1). 与命题 2 对偶的去讨论可知 Δ 只有有限个顶点和箭向. 若在 Δ 中存在循环道路 $x = (a \mid \alpha_1, \cdots, \alpha_n \mid a)$ 及箭向 $\alpha = (a \mid \alpha \mid b), \alpha \neq \alpha_1$,此时易知有下面无限递增右理想链,

$$\langle x\alpha) \subsetneqq \langle x\alpha, x^2\alpha) \subsetneqq \langle x\alpha, x^2\alpha, x^3\alpha) \subsetneqq \cdots$$

因而得证 2)⇒1).

1)⇒2). 设 Δ 的有限个顶点为 $a_1, \cdots, a_n$,而令 $e_i = (a_i \mid a_i)$. 易知 $1 = \sum_i e_i$ 是代数 $K(\Delta)$ 的单位元. 任意右理想 I 可表成

$$I = 1 \cdot I \cdot 1 = \sum_{i,j} e_i I e_j.$$

欲证右理想递增链 $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ 是在有限步终止, 只需证, 对任意 $1 \leqslant i, j \leqslant n$, 下面子代数递增链

$$e_iI_1e_j \subsetneqq e_iI_2e_j \subseteq e_iI_3e_j \subseteq \cdots \tag{1}$$

是在有限步终止即可

说 Δ 中的一个循环道路 $w = (a_j \mid \alpha_1, \cdots, \alpha_t \mid a_j)$ 为极小的，如果箭向 $\alpha_1, \cdots, \alpha_t$ 无相同者. 注意到 Δ 满足条件 ii)，知 Δ 中以 a_j 为起点和终点的循环道路中只有一个极小者，说是 w，而其他的都是形如 w^m 者，并且 Δ 中无有形如 uw^nv 的道路，其中 v 不是 w 的子路且长度 $\geqslant 1$.

$e_iK(\Delta)e_j$ 恰是 $K(\Delta)$ 的元素中可表成形如 $(a_i \mid \alpha_1, \cdots, \alpha_n \mid a_j)$ 的道路的线性和的全体. 形如 $(a_i \mid \alpha_1, \cdots, \alpha_n \mid a_j)$ 的道路可分成二类，一类是不含循环道路为其子道路者，由于 Δ 中只有有限个箭向，这样的道路只有有限个，设为 $u_1, \cdots, u_m$. 另一类是其子道路中有循环的，由上知这些子循环道路都必是以 a_j 为起点和终点的，且可表成 w^n，其中 w 是以 a_j 为起点和终点的唯一极小循环道路. 这样 $e_iK(\Delta)e_j$ 中的元素 x 必可表成下面形状：

$$x = \sum_{i=1}^{m} f_iu_i + \sum_{k\geqslant 1}\Big(\sum_{i=1}^{m} f_i^{(k)}u_i\Big)w^k (\text{有限和}).$$

这样，$e_jK(\Delta)e_j$ 或者同构于 K（当无长度 $\geqslant 1$ 的以 a_j 为起点和终点的循环道路），或者是 $K[w]$，它同构于 K 上一个不定元 y 的多项式代数 $K[y]$.

易见 $e_iK(\Delta)e_j$ 可以看成代数 $e_iK(\Delta)e_j$ 上的右模. 上面的讨论说明：右 $e_jK(\Delta)e_j$ - 模 $e_iK(\Delta)e_j$ 或者是 K 上的有限维向量空间，或者是 K 上多项式代数上的有限生成模. 由模论中熟知结果知 $e_iK(\Delta)e_j$ 是右 Noether $e_jK(\Delta)e_j$ - 模. 显然(1)是右 $e_jK(\Delta)e_j$ - 模 $e_iK(\Delta)e_j$ 的子模升链，因而它必在有限步上终止.

称有向图 Δ 中的一个循环道路 $w = (a \mid \alpha_1, \cdots, \alpha_n \mid a)$ 为孤立的，若 w 所经过的顶点 $a, a_1, \cdots, a_{n-1}$，互不相同且与 Δ 中其余顶点间无箭向相连. 由上命题直接得

命题 4　下列命题等价

1) Δ 只有有限个顶点和有限个箭向且 Δ 中的任意循环道路都是孤立的；

2) $K(\Delta)$ 是左，右 Noether 代数.

这是因为，由命题 3 及其对偶命题知 1)，2) 是等价的.

称有向图Δ中的一个道路$(a \mid \alpha_1, \cdots, \alpha_n \mid b)$, $n \geqslant 1$, 为正则的, 如果它不是任意循环道路的子道路.

命题 5* 1) $K(\Delta)$ 的 Jacobson 根 J 恰是 Δ 中一切正则道路所支持的 K - 向量空间, 因而 J 是诣零理想而与 $K(\Delta)$ 的 Koethe 根重合;

2) $K(\Delta)$ 是半本原代数 $\Leftrightarrow \Delta$ 中无正则道路;

3) $K(\Delta)$ 是半本原代数 $\Leftrightarrow K(\Delta)$ 是半素代数.

这就是说, $K(\Delta)$ 的 Jacobson 根 $=$ $K(\Delta)$ 的 Baer 根.

证明: 1) 说Δ中所有正则道路所支撑的K - 向量空间为M. 首先证$M \subseteq J$. 设 $u = (a \mid \alpha_1, \cdots, \alpha_n \mid b)$ 是任意的一个正则道路. u 在$K(\Delta)$ 中生成的理想为(u). 易知(u) 中元素是一些以 u 为子道路的道路的线性和. 设 x, y 是以u 为子道路的两个道路, 若 $x \cdot y = x_1ux_2 \cdot y_1uy_2 \neq 0$, 其中 x_i, y_i 是道路, 则易见 $ux_2y_1 \neq 0$ 且是一个循环道路并含 u 为其子道路, 这与 u 的正则性相矛盾, 故 $x \cdot y = 0$, 即$(u)^2 = 0$. 但J 是包含一切诣零理想的, 故 $u \in J$, 即 $M \subseteq J$.

今证 $M = J$. 若 $M \neq J$, 则必有$0 \neq x \in J$ 且$x = \sum f_ix_i$(有限和), 其中 $0 \neq f_i \in K$ 而x_i 是非正则道路. 不失一般性, 必要可左乘 x 以e_i 右乘 x 以e_j, 总可设x_i, $\forall_i$, 都是以 a_i 为起点, 以 a_j 为终点者. 若 $a_i \neq a_j$, 由于 x_i 是非正则道路, 即 x_i 是一循环道路的子路, 故必有道路z, 它以 a_j 为起点 a_i 为终点. 由于$0 \neq xz \in J$ 且每一x_iz 都是循环道路, 这样我们不妨认定 J 中有一非零元$x = \sum f_ix_i$, 而所有 x_i 都是以a 为起点和终点的循环道路且 x 的长度$\geqslant 1$, 因为若 x 的长度为零, 则 $x = f(a \mid a)$, 而$(a \mid a)$ 是幂等元, 不可能在J 中. 注意到J 中元都是拟正则元, 故有$0 \neq y \in K(\Delta)$ 使得

$$x + y + xy = 0. \tag{2}$$

显然 y 的长度不能为零. 设 y 的长度$n \geqslant 1$. 令 $y = y_1 + y_2$, 其中出现在 y_1 的道路都是以 a 为起点而出现在y_2 中的道路之起点都异于 a. 此时,

$$x + y + xy = x + y_1 + y_2 + xy_1 = 0$$

注意到 x, y_1, xy_1 中出现的道路都以 a 为起点, 故 $y_2 = 0$, 即 $y = y_1$ 的长度 $n \geqslant 1$, 且

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* 这一结果是首先由肖杰同志得出的.

$$x + y_1 + xy_1 = 0. \tag{3}$$

但此时有 xy_1 的长度 = x 的长度 + y_1 的长度,故(3),因而(2)不可能成立.此矛盾说明必有 $J = M$.证得1)

显然2)是1)的直接推论.为了说明3),设 $K(\Delta)$ 是半素代数,注意证明开始时所得的事实:正则道路 u 生成的理想是幂零的,知此时 Δ 不能有正则道路,随之由2)知,$K(\Delta)$ 是半本原的.另一方向是熟知的命题(参看[2]).

称有向图 Δ 是连接图,如果对任意两不同顶点 a,b,至少有一个以 a 为起点以 b 为终点的道路.

命题6　下列命题等价

1)Δ 是连接图;

2)$K(\Delta)$ 是素代数.

证明:2)⇒1).设 $K(\Delta)$ 是素代数.若 Δ 不是连接图,此时在 Δ 中有两个不相同的顶点 a,b 且 Δ 中没有以 a 为起点以 b 为终点的道路.令 $e = (a \mid a), e' = (b \mid b)$.注意到 $eK(\Delta)e'$ 恰是由 Δ 中以 a 为起点以 b 为终点的道路支撑的 K-向量空间,得 $eK(\Delta)e' = 0$,这与 $K(\Delta)$ 的素性(参看[2])矛盾,即 Δ 必是连接图.

1)⇒2).设 I,J 是 $K(\Delta)$ 的两个非零理想且 $I \cdot J = 0$.由于 $I \neq 0$,故在 I 的非零元素的表达式中必出现以 a 为起点以 b 为终点的路.令 $e_1 = (a \mid a), e_2 = (b \mid b)$,则有 $e_1 I e_2 \neq 0$.同理,有 $e_3 = (c \mid c), e_4 = (d \mid d)$,使 $e_3 J e_4 \neq 0$.由 Δ 的连接性,必有由顶点 b 到 c 的道路 w.这时

$$e_1 I e_2 \cdot w \cdot e_3 J e_4 \neq 0,$$

这与 $I \cdot J = 0$ 矛盾.此矛盾说明 $K(\Delta)$ 是素代数.

称 Δ 的一个循环道路 w 为广义 Hamilton 道路,如 w 至少一次地经过 Δ 的每一个顶点,命题6的另一种表达方式就是:

命题7　Δ 有广义 Hamilton 道路 ⇔ $K(\Delta)$ 是素代数.

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偏序集代数的同构问题*

Isomorphism Problem of Partial Ordered Set Algebras

在[1]中对局部有限偏序集 $I = \{I, \leqslant\}$ 及域 K 引入了关联代数 KI^* 的概念. 这里的"局部有限"是指对任意 $a, b \in I, a \leqslant b$, 集合 $\{x \in I \mid a \leqslant x \leqslant b\}$ 是有限集. KI^* 的定义是:其元素是域 K 上以 I 中元素为行与列的足码的形式矩阵 $(k_{a,b})_{a,b\in I}$(即允许有无限多个 $k_{a,b} \neq 0$)且满足条件:当 $a \not\leqslant b$ 时有 $k_{a,b} = 0$. 注意到 I 的局部有限性, 易知上述形式矩阵的全体关于通常矩阵的加法和乘法以及数乘作成域 K 上的一个结合代数, 称之为 I 在 K 上的关联代数 KI^*. 它是由偏序集 I 作出的一个好的代数系统, 其原因之一是关联代数的同构问题有正面的解答, 即由 $KI_1^* \simeq KI_2^*$ 可得 $I_1 \simeq I_2$, 这里 I_1, I_2 是两个局部有限偏序集(参看[2],[3]).

本文的目的是对任意偏序集引入偏序集代数, 并证明相应的同构问题有正面的解答, 从而偏序集的几何性质和偏序集代数的代数性质之间有相互对应关系, 提供用偏序集代数, 以及关联代数, 去研究偏序集的可能性.

在下面, I 表偏序集, R 表有单位元的环.

定义 1 令 RI 表示 R 上以 I 中元素为行、列足码的矩阵 $(r_{a,b})_{a,b\in I}, r_{a,b} \in R$(只有有限个 $r_{a,b} \neq 0$)且满足条件: $r_{a,b} = 0$ 当 $a \not\leqslant b, \forall a, b \in I$, 的全体. 注意到偏序关系的传递性, 易知 RI 关于矩阵的加、乘运算是封闭的而作成一个结合环. 称环 RI 为偏序集 I 的偏序集代数.

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显然，对于局部有限偏序集 I 言 RI 是 RI^* 的一个子环. 对有限偏序集 I，有 $RI = RI^*$.

虽然本文的目的在于讨论环 R 是域的情形，然而我们的证法对下面较广的一类环也是成立的，为了叙述方便，引入

定义 2 称有单位元 1 的环 R 为唯一幂等元环，如果 1 是 R 的唯一幂等元（在本文中幂等元恒指非零者），且当 R 不是交换环时，R 无幂零元.

例如，域、除环、整区、不可分解交换环等都是唯一幂等元环.

由 RI 的定义易知，矩阵单位 $e_{x,y}, x, y \in I$ 属于环 RI 当且仅当 $x \leqslant y$. 特别 $e_{x,x}, \forall x \in I$，都在 RI 中. 下面叙证中出现的 $e_{x,y}$ 恒指属于 RI 的矩阵单位.

引理 设 $I = \{I, \leqslant\}$ 和 $I' = \{I', \leqslant\}$ 是偏序集而 R 是唯一幂等元环. 设 $f: RI \to RI'$ 是偏序集环间的同构对应，则 $e_{x,x}, x \in I$，在 f 下的象必具有形式：

$$f(e_{x,x}) = e_{x',x'} + \sum_{s' \neq x'}^{(1)} r_{s'x'} e_{s',x'} + \sum_{t' \neq x'}^{(2)} r_{x't'} e_{x',t'} + \sum^{(1)} \cdot \sum^{(2)}.$$

其中 x', s' 等属于 I' 而 $r_{s'x'}$ 等是 R 中元素.

证明：在下面讨论中，I 中元素用 $a, x, \cdots$ 表示，而 I' 中元素用 $a', x', \cdots$ 表示.

首先想说明的是：当 I 是有限偏序集时，记其元素为 $a_1, a_2, \cdots, a_n$ 并可认定它有性质：在 I 中有 $a_i \leqslant a_j$ 时必有 $i \leqslant j$. 这样，RI 便是 R 上的上三角矩阵环的一个子环. 因此，对任意偏序集 I，当考虑 RI 中有限个元素，因而只涉及 I 中有限个元素时，总可以把这些元素都想成是 R 上的一些上三角矩阵.

设 N 是 RI 中一切形如 $re_{x,y}, x \neq y, x, y \in I, r \in R$，以及 $T = \{re_{x,x} \mid x \in I, r \in R$ 且 r 是幂零元$\}$ 生成的子环. 注意到唯一幂等元环的定义，仅当 R 是交换环时，T 中才可能有非零元，但此时 R 中幂零元全体是 R 的一个理想，故知 N 是 RI 的理想且恰是 RI 中一切幂零元的全体. 今证 N 是局部幂零的. 设 S 是任意 n 个形如 $re_{x,x}$，r 是幂零元和任意 m 个形如 $re_{x,y}, x \neq y, r \in R$，的元素集合. 只需证明 S 生成的子环 A 是幂零的即可. 设 S 中 n 个幂零系数 r 的最大幂零指数为 t，则易见子环 A^t 是由

有限个形如 $re_{x,y}, x \neq y, r \in R$ 的元素生成并设其生成元个数为 p. 今从这 p 个形如 $re_{x,y}, x \neq y$ 的生成元中任取 $p+1$ 个(可重复取), 则其乘积必为零, 否则, 注意到 RI 的定义, 必导致 $x < y < x, x, y \in I$, 这在偏序集 I 中是不会出现的. 这样 $(A^i)^{p+1} = 0$, 即 A 是幂零子环. N 的局部幂零性证完.

易见 $RI/N = \bigoplus_{x \in I} \bar{R}\bar{e}_{x,x}$, 其中 $\bar{e}_{x,x}$ 表示 $e_{x,x} + N$, 当 R 不是交换环时 $\bar{R} = R$, 当 R 是交换环时, $\bar{R}$ 是 R 关于其最大诣零理想的商环.

对 RI' 进行同样的讨论, 得 N' 以及 $RI'/N' = \bigoplus_{x' \in I'} \bar{R}\bar{e}_{x',x'}$.

在同构对应 f 下, 幂零元映到幂零元, 故有 $f(N) = N'$. 这样, f 诱导出

$$RI/N = \bigoplus_{x \in I} \bar{R}\bar{e}_{x,x} \to RI'/N' = \bigoplus_{x' \in I'} \bar{R}\bar{e}_{x',x'}.$$

的同构对应 $\bar{f}$.

$\bar{e}_{x,x}$ 是 RI/N 中的本原幂等元, 故 $\bar{f}(\bar{e}_{x,x})$ 也是本原幂等元. 另一方面, 易知在 $\bigoplus \bar{R} \cdot \bar{e}_{x',x'}$ 中本原幂等元必具有形状 $\bar{r}\bar{e}_{x',x'}, \bar{r} \in \bar{R}$, 故有 $\bar{f}(\bar{e}_{x,x}) = \bar{r}\bar{e}_{x',x'}$. 这样 $f(e_{x,x}) \in re_{x',x'} + N$. 适当地把一些项写在一起, 可把 $f(e_{x,x})$ 写成下面形状

$$f(e_{x,x}) = e_{x',x'} + \sum_{s' \neq x'}^{(1)} r_{s'x'}e_{s',x'} + \sum_{t' \neq x'}^{(2)} r_{x't'}e_{x',t'} + \sum^{(1)} \cdot \sum^{(2)} + \sum_{\substack{y' \neq x' \\ z' \neq x'}}^{(3)} r_{y'z'}e_{y',z'} \qquad (*)$$

注意到 $f(e_{x,x})$ 是幂等元而 R 是唯一幂等元环, 故在 $f(e_{x,x})$ 的表达式中 $e_{x',x'}$ 的系数必是 1. 当然在上式中 $\sum^{(1)} \cdot \sum^{(2)}$ 和 $\sum^{(3)}$ 中不排除有同类项.

今证(2) 中 $\sum^{(3)} = 0$. 设在 $(*)$ 中出现的 $r_{a'b'}e_{a',b'}$, 除 $e_{x',x'}$ 外的全体为 S, 而 S 生成的子环为 A. 由上面的讨论知 A 是幂零的, 即有自然数 n 使 $A^n = 0$. 今考察 $(f(e_{x,x}))^n$ 的展开式中的项. 它们都是从 $\{e_{x',x'}\} \cup S$ 中任取 n 个(可重复) 所作的乘积. 注意到 $A^n = 0$, 以及 RI' 中的乘法规则, 易知 $(f(e_{x,x}))^n$ 中不为零的项必含 $e_{x',x'}$ 作为因子, 故只能有以下三种:

$$re_{s',x'} = a \cdot e_{x',x'}^{n-1}; \quad re_{x',t'} = e_{x,x}^{n-1} \cdot \beta;$$
$$re_{s',x'} \cdot e_{x',t'} = a \cdot e_{x',x'}^{n-2} \cdot \beta.$$

其中 α 取自 $\overset{(1)}{\sum}$ 中而 β 取自 $\overset{(2)}{\sum}$ 中. 但 $(f(e_{x,x}))^n = f(e_{x,x})$，而 $f(e_{x,x})$ 中上述类型项恰组成除 $\overset{(3)}{\sum}$ 以外的项，故得 $\overset{(3)}{\sum} = 0$.

下面是本短文的主要结果

定理 1　设 $I = \{I, \leqslant\}$，$I' = \{I', \leqslant\}$ 是两个偏序集，R 是唯一幂等元环，则 $RI \simeq RI'$ 当且仅当 $I \simeq I'$.

证明：　当 $I \simeq I'$ 时，显然有 $RI \simeq RI'$.

下面设 f 是 RI 到 RI' 的同构对应. 由上引理的证明中知 f 诱导出

$$RI/N = \bigoplus_{x\in I}\bar{R}\bar{e}_{x,x} \rightarrow RI'/N' = \bigoplus_{x'\in I'}\bar{R}\bar{e}_{x',x'}$$

的同构对应 $\bar{f}$ 且对任意 $x \in I$ 有 $\bar{f}(\bar{e}_{x,x}) = \bar{r}\bar{e}_{x',x'}$. 作对应 $\theta: x \mapsto x'$，易见 θ 是集合 I 到 I' 上的一个一一对应.

今证 θ 保持关系 $\leqslant$. 若 $x < y$，$x, y \in I$，亦即有 $e_{x,y} \in RI$，即 $e_{x,x} \cdot RI \cdot e_{y,y} \neq 0$. 令 $\theta(x) = x'$，$\theta(y) = y'$，则由引理及其证明知

$$\begin{aligned}
&f(e_{x,x}) \cdot RI' \cdot f(e_{y,y}) \\
&= \Big(e_{x',x'} + \sum_{s'\neq x'}^{(1)} r_{s'x'}e_{s',x'} + \sum_{t'\neq x'}^{(2)} r_{x't'}e_{x',t'} + \overset{(1)}{\sum} \cdot \overset{(2)}{\sum}\Big) \cdot \\
&\quad RI' \cdot \Big(e_{y',y'} + \sum_{l'\neq y'}^{(1)} r_{l'y'}e_{l',y'} + \sum_{m'\neq y'}^{(2)} r_{y'm'}e_{y',m'} + \overset{(1)}{\sum} \cdot \overset{(2)}{\sum}\Big) \neq 0.
\end{aligned}$$

这样必有在 $f(e_{x,x})$ 中出现的 $e_{a',b'}$ 和在 $f(e_{y,y})$ 中出现的 $e_{c',d'}$ 使 $e_{a',b'} \cdot RI' \cdot e_{c',d'} \neq 0$.

若 $e_{x',x'} \cdot RI' \cdot e_{y',y'} \neq 0$. 则有 $e_{x',y'} \in RI'$；

若 $e_{x',t'} \cdot RI' \cdot e_{l',y'} \neq 0$. 则有 $e_{t',l'} \in RI'$，因而 $e_{x',y'} = e_{x',t'} \cdot e_{t',l'} . e_{l',y'} \in RI'$.

其余的七种情形，由类似的计算都必然得 $e_{x',y'} \in RI'$. 这说明在 I' 中有 $x' < y'$.

上述证明方法对由偏序集 I 所确定的下面两类环也是适用的. 我们仅叙述定义和结果而略去证明.

定义 3　偏序集 I 中具最小元 a 和最大元 b 的有序子集 $\alpha = (a, \cdots, b)$ 以及 (x)，$x \in I$，称作 I 的广义路，$a(b)$ 称为 α 的始(终)元而 (x) 的始元终元都是 x. 设 $R[I]^*$ 是以 I 的一切广义路为基的自由 R－模. 今规定它的一个乘法，为此只需规定广义路 α, β 的乘积：$\alpha \cdot \beta = \alpha \cup \beta$，如果

α 的终元等于β 的始元,否则规定 $\alpha \cdot \beta = 0$. 易见 $R[I]^*$ 是一个结合环,称之为偏序集 I 的广义路环. 称 I 的一个有限有序子集为路. I 的所有路在环$R[I]^*$ 中生成的子环,记作 $R[I]$ 并称之为 I 的路环.

定理 2 设 R 是唯一幂等元环, I 和 I' 是偏序集,则有

(1) 广义路环 $R[I]^* \simeq R[I']^*$ 当且仅当 $I \simeq I'$;

(2) 路环 $R[I] \simeq R[I']$ 当且仅当 $I \simeq I'$.

把偏序集解释为有向图, 则上定理中的(2) 可由[4] 中定理直接得出,但这里的证法较简单, 直接.

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偏序集代数的同构问题

Isomorphism Problem of Partial Ordered Set Algebras

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Abstract Let I, I' be arbitrary partially ordered sets and F be a field. We introduce generalized path algebra, path algebra and partially ordered set algebra of I over F. In this paper we give the positive answer of isomorphism problem for these classes of algebras i.e. isomorphism of algebras of the same type, for example $FI \simeq FI'$, induces isomorphism of corresponding partial odered sets I and I'.

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G-分次环与G-集的冲积

Smash Products of G-Graded Rings With G-Set

(SMASH PRODUCT)*①

摘　要　对任意群G,[1]中研究了G-分次环与可迁有限G-集的冲积.在本文中我们对任意可迁G-集A,讨论了G-分次环R与G-集A的冲积,从而推广了[2][3]中给出的关于G-分次环与群G的冲积的主要结果.

关键词　群分次环、G-集、冲积、对偶定理

一、基本定义

在本文中,G表示任意(有限或无限)群,R表示任意有单位元1的G-分次环,即有$R=\bigoplus_{g\in G}R_g$(加群直和)且$R_gR_h\subseteq R_{gh}$,$\forall\, g,h\in G$.G-分次环R上的分次左模范畴记作R-gr.关于群分次环,模的基本性质均参看[4].

设A是G-集,即有映射(G在A上的作用):$G\times A\rightarrow A$满足条件:$g(ha)=(gh)a$,$\forall\, g,h\in G,a\in A$及$ea=a$,$\forall\, a\in A$,其中$e$是群$G$的单位元,易知任意$G$-集$A$都是可迁$G$-集(即对其中任意两元素$x,y$,总有$g\in G$使$gx=y$)的并集,因而我们将局限于只讨论可迁$G$-集.同样易知:任一可迁$G$-集同构于$G$的某个子群$H$在$G$中的全体左陪集$G/H=\{\sigma_iH,i\in I\}$自然作成的$G$-集,其中$\{\sigma_i,i\in I\}$是$H$在$G$中的左陪集的一个完全代表集,$H$在$G$中的指数可为有限或无限.这样,在以

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后的讨论中总把 G - 集取作 $G/H = \{\sigma_i H, i \in I\}$.

设 A 为 G - 集. G - 分次环 R 上的模 M 称作 A 型分次模，如果 $M = \bigoplus_{a \in A} Ma$（加群直和）且有 $R_g M_a \subseteq M_{ga}$, $\forall g \in G, a \in A$. A 型分次模 M 到 A 型分次模 N 的同态 ϕ 称作 A 型分次同态，如果 $\phi(M_a) \subseteq N_a$, $\forall a \in A$. R 上的所有 A 型分次模与它们之间的 A 型分次同态作成的范畴记作 $(A,R) - gr$（参看[1]）.

下面给出 G - 分次环与任意可迁 G - 集的冲积(Smash Product)的定义（参看[1]§2）.

定义 1 设 G 是任意群，R 是 G - 分次环. $G/H = \{\sigma_i H, i \in I\}$ 是任意可迁 G - 集. 对任一 $\sigma_i H$ 引入符号 $P_{\sigma_i H}$. 以 $\{P_{\sigma_i H}, i \in I\}$ 为自由基作左自由 R - 模 $R \# G/H = \bigoplus_{i \in I} RP_{\sigma_i H}$，且在其中定义乘法：

$$\left(aP_{\sigma_i H}\right)\left(bP_{\sigma_j H}\right) = \sum_{\substack{g \in G \\ g\sigma_j H = \sigma_i H}} ab_g P_{\sigma_j H}$$

（再依线性扩张到整个 $R \# G/H$ 上去），其中 $a, b \in R$, $b = \sum_{g \in G} b_g$ 而 $b_g \in R_g$ 是元素 b 的 g - 分量. 由于 G/H 是可迁的，故总有 $g \in G$ 使 $g\sigma_j H = \sigma_i H$. 又由于只有有限个 b_g 不为零，故上式右侧中的求和式是合理的. 直接验证知，加群 $R \# G/H$ 在上述乘法下作成一个结合环，称之为 R 与 G/H 的冲积.

为了简单，以后把元素 $P_{\sigma_i H}$ 和 $1 \cdot P_{\sigma_i H}$ 都记作 P_{σ_i}.

虽然当 G/H 无限时，环 $R \# G/H$ 中没有单位元. 但容易验证 $\{P_{\sigma_i}, i \in I\}$ 是 $R \# G/H$ 的正交幂等元集，且冲积 $R \# G/H$ 局部地具有单位元.

当取 $H = \{e\}$ 而 $G/\{e\} = G$ 时，上述 R 与 G - 集 G 的冲积与[3]中定义的 R 与群 G 的冲积是一致的，因而 R 与 G - 集的冲积是 R 与群 G 的冲积的一个自然推广.

下面将把环 $R \# G/H$ 中的元素用矩阵来表示；从而使该环具有某种矩阵环的形式，这使得其中运算具有矩阵运算的形式，很是方便.

将用 r_g 表示 R_g 中的元素. 当 K 是 G 的子集时，令 $R^K = \sum_{g \in K} R_g$.

由 $R \# G/H$ 中的乘法定义知

$$\left(r_g P_{\sigma_i}\right)\left(r_h P_{\sigma_j}\right) = \begin{cases} (r_g r_h) P_{\sigma_j}, & \text{当 } h_{\sigma_j} H = \sigma_i H \\ 0, & \text{其他情形} \end{cases}$$

所以

$$\left(R_g P_{\sigma_i}\right)\left(R_h P_{\sigma_j}\right)\begin{cases}\subseteq R_{gh} P_{\sigma_j}, & \text{当 } h_{\sigma_j} H = \sigma_i H \\ = 0, & \text{其他情形}\end{cases}$$

故有

$$\left(R^{\sigma_i H \sigma_j^{-1}} P_{\sigma_j}\right)\left(R^{\sigma_l H \sigma_k^{-1}} P_{\sigma_k}\right)\begin{cases}\subseteq R^{\sigma_i H \sigma_k^{-1}} P_{\sigma_k}, & \text{当 } j = l \\ = 0, & \text{当 } j \neq l\end{cases}$$

将环 $R \# G/H$ 表成如下加群直和的形式：

$$\begin{aligned} R \# G/H = & R^{\sigma_i H \sigma_i^{-1}} P_{\sigma_i} + R^{\sigma_i H \sigma_j^{-1}} P_{\sigma_j} + R^{\sigma_i H \sigma_k^{-1}} P_{\sigma_k} \\ & + \cdots + R^{\sigma_j H \sigma_i^{-1}} P_{\sigma_i} + R^{\sigma_j H \sigma_j^{-1}} P_{\sigma_j} + R^{\sigma_j H \sigma_k^{-1}} P_{\sigma_k} \\ & + \cdots + R^{\sigma_k H \sigma_i^{-1}} P_{\sigma_i} + R^{\sigma_k H \sigma_j^{-1}} P_{\sigma_j} + R^{\sigma_k H \sigma_k^{-1}} P_{\sigma_k} \\ & + \cdots\cdots \end{aligned}$$

这样环 $R \# G/H$ 中任一元素 x 可唯一地表成：$x = \sum_{i,j \in I} x_{ij}$，其中 $x_{ij} \in R^{\sigma_i H \sigma_j^{-1}} P_{\sigma_j}$，$\forall\, i, j \in I$. 利用上面的乘法规则可直接验证

$$\phi : x = \sum x_{ij} \mapsto (x_{ij})_{(i,j) \in I \times I} (I \times I \text{ 矩阵})$$

是环同构对应：

$$R \# G/H \simeq \left(R^{\sigma_i H \sigma_j^{-1}} P_{\sigma_j}\right)_{(i,j) \in I \times I}$$

后者是由所有在(i,j)位置上取加群 $R^{\sigma_i H \sigma_j^{-1}} P_{\sigma_j}$ 中元素的$I \times I$ 矩阵(其中只取有限个系数非零)的全体关于通常矩阵的加法乘法作成的环. 另一方面易见有环同构：

$$\left(R^{\sigma_i H \sigma_j^{-1}} P_{\sigma_j}\right)_{(i,j) \in I \times I} \simeq \left(R^{\sigma_i H \sigma_j^{-1}}\right)_{(i,j) \in I \times I},$$

其中右侧矩阵环的定义与左侧的一样，这样便有

$$R \# G/H \simeq \left(R^{\sigma_i H \sigma_j^{-1}}\right)_{(i,j) \in I \times I}.$$

这就是我们所需要的冲积的矩阵表示. 今后将用矩阵环(简记作)$\left(R^{\sigma_i H \sigma_j^{-1}}\right)$来代替冲积$R \# G/H$.

二、主要结果

下面定理把 G-分次环R 上的G/H 型分次模转化为环$R \# G/H$ 上

的模.

定理 1 范畴$(G/H,R)-gr$与范畴$R\# G/H-\mathrm{mod}$(指所有满足$(R\# G/H)M=M$的左$R\# G/H$-模M组成的范畴)是同构的:$R\# G/H-\mathrm{mod}\simeq(G/H,R)-gr$.

证明:任取$(G/H,R)-gr$中的模M,则依定义有$M=\bigoplus_{i\in I}M_{\sigma_iH}$(加群直和)且$R_g\cdot M_{\sigma_iH}\subseteq M_{g\cdot\sigma_iH}$.作$I\times 1$矩阵(即以$I$中元素命名行)加群

$$M_*=\left(M_{\sigma_iH},M_{\sigma_jH},M_{\sigma_kH},\cdots\right)^T,$$

(即在$(i,1)$位置取M_{σ_iH}中元素,而每一$I\times 1$矩阵中只有有限个系数非零)规定$R\# G/H=(R^{\sigma_iH\sigma_j^{-1}})$与$M_*$之间的乘法为通常$I\times I$矩阵与$I\times 1$矩阵的乘法,而矩阵系数之间的运算则按$R$-模$M$中的运算进行.易知此时$M_*$作成一左$R\# G/H$-模且有$(R\# G/H)M_*=M_*$.若$f:M\to N$属于$(G/H,R)-gr$,则它自然给出$M_*$到$N_*$的对应$f_*(=f)$.直接验证可知$f_*$是$R\# G/H$-模同态,且

$$(\quad)_*:\quad(G/H,R)-gr\to R\# G/H-\mathrm{mod}$$
$$M\mapsto M_*$$
$$f\mapsto f_*$$

是加法共变函子.反之,取$N\in R\# G/H-\mathrm{mod}$,则$N=(R\# G/H)N=\bigoplus_{i\in I}P_{\sigma_i}N$.令$N^*=N=\bigoplus_{i\in I}N^*_{\sigma_iH}$(加群直和),其中$N^*_{\sigma_iH}=P_{\sigma_i}N$.规定$R$与$N^*$之间的乘法为

$$r_g\cdot n_{\sigma_iH}=\left(r_gP_{\sigma_i}\right)n_{\sigma_iH},\ r_g\in R_g,\ n_{\sigma_iH}\in N^*_{\sigma_iH}$$

再依线性扩张之.因为

$$\left(r_gP_{\sigma_i}\right)n_{\sigma_iH}=\left(1P_{g\sigma_iH}\right)\left(r_gP_{\sigma_iH}\cdot n_{\sigma_iH}\right)\subseteq N^*_{g\sigma_iH},$$

故$N^*\in(G/H,R)-gr$.若$g:N\to M$属于$R\# G/H-\mathrm{mod}$,令$g^*=g$,则$g^*:N^*\to M^*$属于$(G/H,R)-gr$.易验证

$$(\quad)^*:\quad R\# G/H-\mathrm{mod}\to(G/H,R)-gr$$
$$N\mapsto N^*$$
$$g\mapsto g^*$$

是加法共变函子,且

$$(M_*)^*=M,(f_*)^*=f,\ \forall M,f\in(G/H,R)-gr,$$

$$(N^*)_* = N, (g^*)_* = g, \forall N, g \in R \# G/H - \text{mod},$$

所以( )$_*$ 与( )* 是互逆的同构函子.

下面我们给出类似于 Cohen-Montgomery 对偶性定理(参看[2])的结果. 先述一个将用的

定理 2 (见[3], 引理 2.1) 令 R 是环, 不一定有单位元. 若 R 有一个完全矩阵单位系 $\{e_{i,j}, i,j \in I\}$, 即有

$$R = \sum_{i,j\in I} e_{i,i} R e_{j,j} \text{ 且 } e_{i,j} e_{l,k} = \begin{cases} e_{i,k}, & \text{当 } j = l \\ 0, & \text{当 } j \neq l \end{cases}$$

$\forall i,j,l,k \in I$, 则 $R \simeq S_{|I|}$. 其中 $S = e_{1,1} R e_{1,1}$, $1 \in I$ 而 $S_{|I|}$ 表示系数取自 S 的所有 $I \times I$ 矩阵(即以 I 中元素命名行列且只有有限个系数非零的矩阵) 组成的矩阵环.

定理 3 设 S 是有单位元的环, 群 G 作用于环 S 上, 即有群同态 φ: $G \to \text{Aut } S$. 作斜群环 $S * G$, 即它是以 G 为基的左自由 S - 模, 其乘法为 $(sg)(th) = st^{g^{-1}} gh$, $s,t \in S$, $g,h \in G$. 对 G - 分次环 $S * G$, 我们有 $(S * G) \# G/H \simeq (S * H)_{|G/H|}$.

证明: 令 $e_{i,j} = 1(\sigma_i \sigma_j^{-1}) P_{\sigma_j}$. 今证 $\{e_{i,j}, i,j \in I\}$ 是 $(S * G) \# G/H$ 的一个完全矩阵单位系. 首先有

$$e_{i,j} \cdot e_{l,k} = (1\sigma_i\sigma_j^{-1}) P_{\sigma_j} \cdot (1\sigma_l\sigma_k^{-1}) P_{\sigma_k}$$

$$= \begin{cases} (1\sigma_i\sigma_j^{-1})(1\sigma_j\sigma_k^{-1}) P_{\sigma_k} = (1\sigma_i\sigma_k^{-1}) P_{\sigma_k} = e_{i,k}, & \text{当 } j = l \\ 0, & \text{当 } j \neq l \end{cases}$$

另一方面, 我们有: 当 $s \in S, g \in G$

$$e_{i,i} \cdot (sg) P_{\sigma_k} = (1e) P_{\sigma_i} \cdot (sg) P_{\sigma_k} = \begin{cases} (sg) P_{\sigma_k}, & \text{当 } g\sigma_k H = \sigma_i H \\ 0, & \text{其他情形} \end{cases}$$

$$(sg) P_{\sigma_k} \cdot e_{i,i} = (sg) P_{\sigma_k} \cdot (1e) P_{\sigma_i} = \begin{cases} (sg) P_{\sigma_k}, & \text{当 } k = i \\ 0, & \text{其他情形} \end{cases}$$

所以 $(S * G) \# G/H = \sum_{i,j} e_{i,i} (S * G) \# G/H \cdot e_{j,j}$. 这样环 $(S * G) \# G/H$ 有一个完全矩阵单位系, 依引理 2 得: 它同构于矩阵环 $S'_{|G/H|}$. 若取 $\sigma_1 = e$, $1 \in I$, 则 $S' \simeq e_{1,1}((S * G) \# G/H)_{e_{1,1}} \simeq S * H$.

上面定理是讨论先作斜群环后作冲积的情形. 还有一种情形是先作冲积再作斜群环, 为此我们需要定义群 G 对冲积 $R \# G/H$ 的作用. 自然

的定义是:$(rP_{\sigma_i H})^g = (rP_{g^{-1}\sigma_i H})$,然而当群 G 是不交换时,一般言,它不是环的自同构(不保持乘法).这是下面定理只考虑 G 是交换群的原因:当 G 是交换群时,定义 $(rP_{\sigma_i H})^g = (rP_{g\sigma_i H})$,直接验证知,它是环 $R\# G/H$ 的自同构而得 G 到该环上的作用,依之可作斜群环 $(R\# G/H)*G$.

定理 4 设 G 是交换群,按上面定义群 G 到 $R\# G/H$ 上的作用.我们有 $(R\# G/H)*G\simeq S_{|G/H|}$,其中 $S\simeq\bigoplus_{j\in I}(R^{H\sigma_j^{-1}}P_{\sigma_j}*\sigma_j H)$.

证明: 令 $e_{i,j}=(1P_{\sigma_i})*\sigma_i^{-1}\sigma_j$ 而来验证 $\{e_{i,j}, i,j\in I\}$ 是环 $(R\# G/H)*G$ 中的一个完全矩阵单位系.首先有

$$\begin{aligned}e_{i,j}\cdot e_{k,l} &= (1P_{\sigma_i}*\sigma_i^{-1}\sigma_j)(1P_{\sigma_k}*\sigma_k^{-1}\sigma_l)\\ &= (1P_{\sigma_i H})(1P_{(\sigma_i^{-1}\sigma_j^{-1})\sigma_k H})*\sigma_i^{-1}\sigma_j\sigma_k^{-1}\sigma_l\\ &= \begin{cases}(1P_{\sigma_i})*\sigma_i^{-1}\sigma_l = e_{i,l}, & \text{当 } j=k\\ 0, & \text{当 } j\neq k.\end{cases}\end{aligned}$$

为了验证

$$(R\# G/H)*G=\bigoplus_{i,j}e_{i,i}\cdot(R\# G/H)*G\cdot e_{j,j}, \tag{1}$$

我们计算如下:约定 $h\in H, g\in G, e$ 是 G 的单位元而用 r_g 表示 R_g 中的元素,有

$$\begin{aligned}e_{i,i}((r_{\sigma_i h\sigma_j^{-1}}P_{\sigma_j})*g) &= (1P_{\sigma_i}*e)\cdot((r_{\sigma_i h\sigma_j^{-1}}P_{\sigma_j})*g)\\ &= (1P_{\sigma_i})(r_{\sigma_i h\sigma_j^{-1}}P_{\sigma_j})*eg\\ &= (r_{\sigma_i h\sigma_j^{-1}}P_{\sigma_j})*g\end{aligned}$$

最后一个符号是因为 $\sigma_i H=\sigma_i h\sigma_j^{-1}\cdot\sigma_j H$.这样,$e_{i,i}$ 是 $R^{\sigma_i H\sigma_j^{-1}}P_{\sigma_j}*G$ 的左单位元.另一方面,令 $g\cdot\sigma_i H=\sigma_j H$(这在下面第二个等号中用到),有

$$\begin{aligned}((rP_{\sigma_j})*g)e_{i,i} &= ((rP_{\sigma_j})*g)(1P_{\sigma_i}*e) = (rP_{\sigma_j})(1P_{\sigma_j})*ge\\ &= (rP_{\sigma_j})*g.\end{aligned}$$

所以 $e_{i,i}$ 是 $RP_{\sigma_j}*g$,其中 $g\cdot\sigma_i H=\sigma_j H$ 的右单位元.注意到

$$(R\# G/H)*G=\bigoplus_{i,j,g}R^{\sigma_i H\sigma_j^{-1}}P_{\sigma_j}*g,$$

故得(1),依引理 2,有

$$(R\# G/H)*G\simeq S_{|G/H|},$$

其中,若取 $\sigma_1=e$(G 的单位元),$1\in I$,则

$$S \simeq e_{1,1} \cdot ((R \# G/H) * G) \cdot e_{1,1} = \oplus_{j,g}(R^{eH\sigma_j^{-1}} P_{\sigma_j} * g) \cdot e_{1,1}$$
$$= \otimes_{j \in I, h \in H}(R^{H\sigma_j^{-1}} P_{\sigma_j} * \sigma_j h) = \otimes_j R^{H\sigma_j^{-1}} P_{\sigma_j} * \sigma_j H.$$

现在我们回到一般情形, 即群 G 可以是不交换的. 由冲积的矩阵环表示, 有

$$R \# G/H \simeq (R^{\sigma_i H\sigma_j^{-1}})_{I\times I} = \oplus_{h\in H}(R^{\sigma_i h\sigma_j^{-1}})_{I\times I}(\text{加群直和})$$

若令$(R \# G/H)_h = (R^{\sigma_i h\sigma_j^{-1}})_{I\times I}$, 则

$$(R \# G/H) = \oplus_{h\in H}(R \# G/H)_h$$

且$(R \# G/H)_h \cdot (R \# G/H)_l \subseteq (R \# G/H)_{hl}$,

其中 h, l 是H 中任意元素. 这样环 $R \# G/H$ 自然地可解释为群H - 分次环(没有单位元的). 如果说 G - 分次环R 与群G 的冲积$R \# G$ 是完全的非分次化, 则 G - 分次环R 与G - 集G/H 的冲积$R \# G/H$ 是部分非分次化, 即将 G - 分次环R 转变成H - 分次环$R \# G/H$, 而这里的 H 是群G 的子群. 下述定理给出了非分次化过程的一种传递关系.

定理5 设H, K 是G 的子群且$K \subseteq H \subseteq G$. 令 G - 集$G/H = \{\sigma_i H, i \in I\}$, H - 集$H/K = \{\delta_l K, l \in P\}$, 因而 G - 集$G/K = \{\sigma_i\delta_l K, i \in I, l \in P\}$. 则有$(R \# G/H) \# H/K \simeq R \# G/K$(上面定义 $R \# G/H$ 时并未用到 R 中单位元, 因而$(R \# G/H) \# H/K$ 是有意义的).

证明: 令

$$\phi:\ (R \# G/H) \# H/K \to R \# G/K$$
$$(rP_{\sigma_i})P_{\delta_l} \mapsto rP_{\sigma_i\delta_l}$$

再依线性扩张到整个环上去. 显然 ϕ 是加群同构. 为了验证 ϕ 也保持乘法, 我们计算如下: $r, s \in R$; $i, j \in I$; $l, m \in P$; 当 x 是群T - 分次环中的元素时, 用 $x_t, t \in T$, 表示 x 的t - 齐次分量.

$$((rP_{\sigma_i})P_{\delta_l}) \cdot ((sP_{\sigma_j})P_{\delta_m})$$
$$= \sum_{\substack{h \in H\text{且} \\ h\cdot\delta_m K = \delta_l K}} ((rP_{\sigma_i})(sP_{\sigma_j})h)P_{\delta_m}$$
$$= \sum_{\substack{h \in \delta_l K\delta_m^{-1} \\ k \in I}} ((rP_{\sigma_i})(s_{\sigma_k h\sigma_j^{-1}}P_{\sigma_j}))P_{\delta_m}$$
$$= \sum_{h \in \delta_l K\delta_m^{-1}} ((rs_{\sigma_i h\sigma_j^{-1}})P_{\sigma_j})P_{\delta_m}.$$

$$= \sum_{w \in K} ((rs_{\sigma_i \delta_l w \delta_m^{-1} \sigma_j^{-1}}) P_{\sigma_j}) P_{\delta_m},$$

在计算第三个等号时我们用到 $h \in H$. 另一方面,

$$(rP_{\sigma_i \delta_l})(sP_{\sigma_j \delta_m}) = \sum_{\substack{g \in G \text{且} \\ \sigma_i \delta_l K = g \cdot \sigma_j \delta_m K}} (rs_g) P_{\sigma_i \delta_m}$$

$$= \sum_{w \in K} (rs_{\sigma_i \delta_l w \delta_m^{-1} \sigma_j}) P_{\sigma_j \delta_m},$$

故得 ϕ 是环同构.

既然由 G - 分次环 R(有单位元) 可得到另一个 H - 分次环 $R \# G/H$(一般没有单位元), 自然要问: 范畴 R - gr 和 $R \# G/H$ - gr 之间有什么关系. 易见, 当 G/H 是有限集, 随之 $R \# G/H$ 是有单位元的 H - 分次环, 则由前面定理 I 及其推论和定理 5 可得

$$R \# G/H\text{-}gr \simeq ((R \# G/H) \# H)\text{-mod} = R \# G\text{-mod},$$

$$R\text{-}gr \simeq R \# G\text{-mod}$$

故有 R - $gr \simeq R \# G/H$ - gr, 当 G/H 是无限集时, 我们也有同样结果, 为此需将定理 1 推广如下.

定理 6 设 T 是 G - 分次环(不一定有单位元) 而 $\{1_i, i \in I\}$ 是 T 的一个正交幂等元系, 且是完全的(即 $T = \sum_{i,j} 1_i T 1_j$), 则有 T - $gr \simeq T \# G$-mod(对没有单位元的环 S 只考虑 $S \cdot M = M$ 的模 M).

证明: 由于 $T = \sum 1_i T 1_j$, 故

$$T \# G = \bigoplus_{h, g \in G}^{i, j \in I} (1_i T_{hg^{-1}} 1_j) P_g (\text{加群直和})$$

而其元素 x 可唯一地写成 $x = \sum (1_i t_{hg^{-1}} 1_j) P_g$. 令

$$M_{|I \times G|} \equiv (1_i T_{hg^{-1}} 1_j p_g)_{(i \times h, j \times g) \in (I \times G) \times (I \times G)}$$

后者表示以 $I \times G$ 中元素命名行列, 在 $(i \times h, j \times g)$ 位置上取加群 $1_i T_{hg^{-1}} 1_j P_g$ 中的元素而得到的所有矩阵(只有有限个系数非零) 依矩阵加法乘法作成的矩阵环. 作映射

$$\phi: T \# G \to M_{|I \times G|}$$

$$x = \sum (1_i t_{hg^{-1}} 1_j) P_g \mapsto (1_i t_{hg^{-1}} 1_j P_g)$$

直接计算知 ϕ 是环同构. 另一方面, 我们看到在矩阵的系数中出现的 P_g 只起定位作用而在矩阵的运算中丝毫没有作用, 故可得环同构

$$M_{|I \times G|} \simeq (1_i T_{hg^{-1}} 1_j)_{(i \times h, j \times g) \in (I \times G) \times (I \times G)}$$

$$(1_i t_{ng^{-1}} 1_j P_g) \mapsto (1_i t_{ng^{-1}} 1_j)$$

故有

$$T\#G \simeq (1_i T_{hg^{-1}} 1_j)_{(i\times h, j\times g)\in(I\times G)\times(I\times G)}$$

这样我们得冲积 $T\#G$ 的矩阵形式，下面利用它建立 $T\text{-}gr$ 和 $T\#G\text{-mod}$ 之间的同构函子.

取 $M\in T\text{-}gr$，则 $M=\oplus_g M_g$. 利用 $M=TM$ 及 $T=\sum_i 1_i T$，得 $M=\oplus_{i,g}1_i M_g$. 令

$$(\quad)_*:\qquad T\text{-}gr \to T\#G\text{-mod}$$

$$M\mapsto M_* = (1_i M_g)_{(i\times g,1)\in(I\times G)\times\{1\}}$$

$$f:\qquad M\to N\mapsto f_* = f: M_*\to N_*,$$

其中 M_* 是所有列向量，在 $(i\times g,1)$ 位置上取 $1_i M_g$ 中元素所作成的加群，$T\#G$ 对 M_* 的模运算定义为 $(I\times G)\times(I\times G)$ 矩阵对 $(I\times G)\times\{1\}$ 矩阵乘法而系数间的乘法依 T－模 M 的模运算进行，$f_*=f$ 的意义是：

$$f^*(1_i m_g)_{(I\times G)\times\{1\}} = (f(1_i m_g))_{(I\times G)\times\{1\}}.$$

直接计算知 $(\quad)_*$ 是正变加法函子.

另一方面，取 $N\in T\#G\text{-mod}$. 利用 $T\#G$ 中的完全正交幂等元系 $\{1_iP_g, i\in I, g\in G\}$ 得 $N=\oplus_{i,g}(1_iP_g)N$. 令

$$(\quad)^*:\qquad T\#G\text{-mod}\to T\text{-}gr$$

$$N\mapsto N^* = N = \oplus_g(N^*)_g,\ \text{其中}(N^*)_g = \oplus_i(1_ip_g)N,$$

$$h:\qquad N\to M\mapsto h^* = h: N^*\to M^*$$

其中把 T 对 N^* 的模运算定义为：$t_h\cdot(1_iP_g)n = ((t_h1_i)P_g)n$，这里 $t_h\in T_h, n\in N, h\in G$. 由于 t_h 有左单位元 $\sum_j 1_j$（有限和），故

$$((t_h1_i)P_g)n = ((\sum 1_j)t_h1_i)P_gn = ((\sum 1_j)P_{hg}\cdot(t_h1_j)P_g)n$$

$$= (\sum 1_j)P_{hg}\cdot((t_h1_j)P_g\cdot n)\in(N^*)_{hg}$$

即 $T_h\cdot(N^*)_g\subseteq(N^*)_{hg}$，故 $N^*\in T\text{-}gr$. 直接计算知 $(\quad)^*$ 是正变加法函子，且有

$$(M_*)^* = M,\ (f_*)^* = f,\ \forall M,\ f\in T\text{-}gr,$$

$$(N^*)_* = N,\ (h^*)_* = h,\ \forall N,\ h\in T\#G\text{-mod}$$

所以 $(\quad)_*$ 与 $(\quad)^*$ 是互逆的同构函子.

有了定理 6，再重复在它前面的讨论，便得

定理 7　R 是有单位元的 G - 分次环, H 是群 G 的子群, 把 $R\# G/H$ 看成 H - 分次环时, 则有 $R\# G/H\text{-}gr \simeq R\text{-}gr$.

把定理 1 和定理 7 放在一起, 使我们具体地看到同一个环 $R\# G/H$ 的模范畴与其分次模范畴之间的差别. 对 G - 分次环 R 与其冲积 $R\# G/H$(看作环, 或看作 H - 分次环) 之间其他关系, 例如它们的 Jacobson 根之间的关系, 我们将在另文讨论.

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在中国有关根论的最近研究工作

Recent Research Work on Radicals in China

From 1978, we resumed the research work on ring theory and introduced some young algebraists to work on radicals. Here I would like to give a brief introduction to some of these works so as to give a general idea of the research work on radicals in China.

1. Structure Theorems of rings

Wedderburn-Maltsev structure theorems of finite dimensional algebras, as well known, occupy a central position and E. Artin has given an excellent generalization. There is another way for their generalization, that is, to extend these theorems to locally finite algebras. It is proved in [28], [29], [30] that we also have structure theorems of Wedderburn-Maltsev type for locally finite subideal (associative, alternative, Lie and Jordan) algebras, where the locally finite subideal algebra is defined to be an algebra which has a local system of finite dimensional local subideals, that is, finite dimensional subideals each of which is an ideal of every subalgebra generated by a finite subset of the algebra and itself. We proved these theorems once for all the four classes instead of proving them separately.

2. Concrete radical constructions for rings

It is natural to generalize concrete radicals of associative rings to some important non-associative rings. In [27] the existence of the Levitzki radical for Lie rings with Engel's conditions and for Jordan rings has been

established. For the case of Jordan rings, this result was proved independently by E. K. Zhevlakov [50].

In 1966, Sulinski, Anderson and Divinsky raised an interesting problem of finding a class $\mathbf{L}^{(n)}$ of associative rings such that the lower radical construction for $\mathbf{L}^{(n)}$ terminates at exactly n + 1 steps. In [17] Guo Jinyun has found such examples. Let F be the field of p elements, $T = (x)$ be the ideal generated by x in the ring $F[x]$ of polynomials. T has an accessible subring (subideal) $B(n)$ of the form $B(n) = \{\alpha(x) + \beta(x) \in F[x]\}$ where $\alpha(x)$ is of the form $\sum_{s=1}^{n-1} \alpha_s x^{ns}$ and $\beta(x)$ of the form $\sum_{t=0}^{f} \beta_t x^{n^2+t}$. Take $\mathbf{K}^{(n)} = \{B(n), \mathbf{Z}^\circ\}$, where $\mathbf{Z}^\circ$ is the zero ring of the infinite cyclic group and let $\mathbf{P}$ be the set of all rings A having a homomorphic image $\phi(A)$ satisfying $B(n) \subseteq \phi(A) \subseteq T$. Then $\mathbf{L}^{(n)} = \{(\mathbf{K}^{(n)**} \setminus \mathbf{P} \cup \mathbf{K}^{(n)}\}$ is the desired class of rings for every $n \geqslant 3$($\mathbf{K}^{(n)**}$ denotes the lower radical determined by $\mathbf{K}^{(n)}$.

After this work was done, we noticed that K. Beĭdar had also given examples for this problem [1].

F. A. Szasz raised the problem of whether the class of biregular rings is a radical class ([32]). In [18], Guo has found an example to show that this is not the case. Let F be a field, $A_1 = F$ and $A_i = F_n$ for some $n \geqslant 2$, for all $i \geqslant 2$. In the ring

$$A = \{f \in \prod_{i=1}^{\infty} A_i \mid f(i) = \begin{pmatrix} f(1) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

for all but a finite number of i's}.

$B = \{fA \mid f(1) = 0\}$ is the maximal biregular ideal of A and $A/B \simeq F$ is also a biregular ring.

3. Characterization of ring-radicals by classes of modules

Andrunakievich and Ryabuhin have given module-theoretic characterizations of radicals and special radicals. Zhou Yiqiang and Li Huishi

have done the work for supernilpotent radicals along the same lines ([24], [51], [52]). They introduced the more general concept of a general property of modules and a general class of modules.

A property **E** of modules (an A-module with property **E** will be called an A-**E** module) is said to be a general property, if **E** satisfies the following conditions:

(E 1) If M is an A-**E** module, then $M \neq 0$ and $xA \neq 0$ for every $0 \neq x \in M$;

(E 2) If M is an A-**E** module, B is an ideal of A and $MB \neq 0$, then $\overline{M} = M/N$ is a B-**E** module, where $N = \{x \in M \mid xB = 0\}$. And in this case $(0:M)_A = 0$ implies $(0:\overline{M})_B = 0$;

(E 3) If B is an ideal of A, M is a B-**E** module, then MB is an A-**E** module and $(0:MB)_A \cap B = (0:M)_B$;

(E 4) Let T be an ideal of A, $\overline{A} = A/T \neq 0$. Then every A-**E** module M with $MT = 0$ is an A-**E** module with scalar multiplication: $m\bar{a} = ma$, $m \in M, a \in A$.

A class $\mathbf{M} = \bigcup_A \mathbf{M}_A$ of modules is called a general class of modules with respect to a general property **E**, if **M** satisfies the following conditions:

(M 1) If $M \in \mathbf{M}_A$, then M is an A-**E** module;

(M 2) If B is an non-zero ideal of A, $M \in \mathbf{M}_A$ and $(0:M)_A = 0$, then $\overline{M} = M/N \in \mathbf{M}_B$, where $N = \{x \in M \mid xB = 0\}$;

(M 3) If $M \in \mathbf{M}_B$ and B is an ideal of A, then $MB \in \mathbf{M}_A$;

(M 4) If T is an ideal of A and $\overline{A} = A/T \neq 0$, then $M \in \mathbf{M}_A$ and $MT = 0$ (as in $(E4)$ M may be considered as $\overline{A}$-module) imply $M \in \mathbf{M}_{\overline{A}}$, and conversely $M \in \mathbf{M}_{\overline{A}}$, (as in (E4) M may be considered as A-module) implies $M \in \mathbf{M}_A$.

A ring A is called an **E**-ring with respect to a general property E if A has a faithful A-**E** module. A class **K** of rings is called a **E**-class of rings if **K** satisfies the following three conditions (1) K consists of **E** rings; (2) **K** is hereditary; (3) if A has an ideal $B \in \mathbf{K}$ such that $\mathrm{Ann}_A B = 0$, then $A \in \mathbf{K}$.

Zhou Yiqiang and Li Huishi established the connection between the upper radical class determined by an **E**-class of rings and the general class of **E**-modules in the following two theorems ($\mathbf{M}(A)$ denotes $\bigcap (0:M)_A \mid M \in \mathbf{M}_A$):

Theorem A. If $\mathbf{M}$ is a general class of **E**-modules and $R(\mathbf{M}) = \{$rings $A \mid \mathbf{M}(A) = A\}$, $S = \{$rings $A \mid \mathbf{M}(A) = 0\}$, $M(\mathbf{M}) = \{$rings $A \mid \exists M \in \mathbf{M}_A$ such that $(0:M)_A = 0\}$, then

1. $M(\mathbf{M})$ is an **E**-class of rings,

2. $R(\mathbf{M})$ is the upper radical class determined by $M(\mathbf{M})$ and $R(\mathbf{M})$ is hereditary,

3. S is the $R(\mathbf{M})$-semisimple class of rings and for an arbitrary ring A, $A \in S$ if A is a subdirect sum of rings taken from $M(\mathbf{M})$.

Theorem B. If $\mathbf{K}$ is an **E**-class of rings, then there is a general class of **E**-modules $\mathbf{M} = \bigcup \mathbf{M}_A$, where
$\mathbf{M}_A = \{$A-modules $M \mid A/(0:M)_A \in \mathbf{K})\}$, such that $\mathbf{K} = M(\mathbf{M})$. When specializing to the semiprime module property these theorems provide a module theoretic characterization of supernilpotent radicals. (An A-module M is called an A-semiprime module if (1) $M \neq 0$ and $xA \neq 0$ for every $0 \neq x \in M$, (2) if $x \in M$ and I is an ideal of A, then $xI^n = 0$ implies $xI = 0$ for every $n \geqslant 1$). It also has been proved that the property of being a prime module is a general property. Other general properties are given in [24].

Li Chuanhe has proved in [22] the analogue for hereditary radicals and hereditary classes of modules, the latter being defined to be classes $\mathbf{M} = \bigcup_A \mathbf{M}_A$ of modules satisfying the following conditions: (1) $M \in \mathbf{M}_A$, then $MA \neq 0$; (2) ($M4$); (3) $M \in \mathbf{M}_B$ if $M \in \mathbf{M}_A$ and $MB \neq 0$ for every nonzero ideal B of A; (4) If $M \in \mathbf{M}_B$, then there is an $\overline{M} \in \mathbf{M}_A$ such that $B \cap (0:\overline{M})_A = (0:M)_B$ for every nonzero ideal B of A.

4. MHR-rings

MHR-rings (rings with minimal condition on principal right ideals) are generalizations of Artinian rings. Xi Changchang, Zhu Xiaozhang and Cai Changren have done some work to solve problems concerning MHR-rings

posed by Szász in [32]. They proved that every subring of a nil MHR-ring is a nil MHR-ring, that every nil subring is locally nilpotent and that every nil MHR-ring is a **Z**-radical ring ([36], [53]). These results solve problems 29,43 and 37 in [32], respectively. Moreover, the same authors gave the following example to solve problems 30 and 48 of [32]:

Let F be a finite prime field, A an algebra over F with base $X = \{e_{ij} \mid i, j \in \mathbf{N}, i > j\}$ and multiplication given by the equations

$$e_{ij} e_{kl} = \delta_{jk} e_{il}$$

where δ_{jk} is the Kronecker-delta. It is shown that A is a nil MHR-ring with an infinite descending chain of principal left ideals

$$(e_{21} > \supset (e_{31} > \supset \cdots$$

($(e_{kl} >$ denotes the left ideal generated by e_{kl}).

In [53] it is also proved that $B = \{x \in A \mid xA = 0\}$ is a non-trivial quasi-modular right ideal of A; A contains no maximal quasi modular right ideal since A is nil and hence contains no modular right ideals. This shows that B cannot be embedded into a maximal quasi-modular right ideal and solves problem 48 of [32] negatively.

5. Γ-rings

There are four concepts of nilpotency in Γ-rings and it is interesting to study their interrelations. Chen Weixin and Guo Yuanchun proved that any of the following conditions is sufficent for every strongly nil subring of a Γ-ring R to be strongly nilpotent.

(1) Maximum condition for strongly nilpotent subrings;

(2) Ascending chain condition for strongly nilpotent subrings;

(3) Noetherian conditions;

(4) Goldie's conditions;

(5) Ascending chain conditions for both left and right annihilators;

(6) Ascending chain condition for left zero-divisor ideals, where a left zero-divisor ideal is defined to be a left ideal I of R such that $I\text{-}\Gamma x = 0$ for some $x \in \mathbf{R}$;

(7) Almost Noetherian condition for left ideals, that is, there is a $p \in$

N such that $(R\Gamma)^p L_i \subseteq L_p$ for every ascending chain

$$L_1 \subseteq L_2 \subseteq \cdots$$

of left ideals of R.

(8) Almost Artinian condition for left ideals, that is, there exists a $g \in$ **N** such that $(R\Gamma)^g L \subset L_i$ for every descending chain

$$L_1 \supseteq L_2 \supseteq \cdots$$

of left ideals of R. ([10], [19], [49])

In [12], [13] Chen even proved the following stronger results:

If a Γ-ring R satisfies one of the following conditions

(1) R is Goldie

(2) R is Noetherian

(3) has the ascending chain condition for left α-zero divisor ideals $l_a(\chi) = \{y \in R \mid y\alpha\chi = 0\}$ then every nilsubring of R is strongly nilpotent.

Chen has also proved the existence of the Von Neumann regular radical in Γ-rings and weakly Γ_n-rings and investigated properties of this radical ([8], [11]).

In [43] Xu Zhongming has generalized Jacobson's density theorem to primitive Γ-rings. Applying his theorem to Artinian Γ-rings, he gets the following result:

Let Λ be the right operator ring of a Γ-ring R, then R is a simple and right Artinian Γ-ring if and only if Λ is a simple right Artinian ring and $A_R(R) = \{x \in R \mid R\Gamma x = 0\} = 0$. In [42] Xu extended classical structure theorems to right Artinian Γ-rings.

6. Other generalizations of rings

In [39] Xu introduced the concept of a Γ-hemiring defined as two Abelian semigroups Γ and S with zeros together with an operation $S x \Gamma x S \to S(x, \alpha, y) \to x\alpha y \in S$ satisfying:

(1) $(x\alpha y)\beta z = x\alpha(y\beta z)$;

(2) $(x + y)\alpha z = x\alpha z + y\alpha z, x(\alpha + \beta)y = x\alpha y + x\beta y$;

(3) $x\alpha y = 0$ if and only if any of them is zero.

Following [2], [3] Xu defined a Jacobson radical $J(S)$ and the semiradical $\sigma(S)$ of a Γ-hemiring S as follows: a right ideal I of S is said to be right quasi-regular if for any $i_1, i_2 \in I$ and any $\alpha \in \Gamma$ there exist $j_1, j_2 \in I$ such that

$$i_1 + j_1 + i_1\alpha j_1 + i_2\alpha j_2 = i_2 + j_2 + i_1\alpha j_2 + i_2\alpha j_1.$$

The sum $J(S)$ of all right quasi-regular right ideals is called the right Jacobson radical of S. Introducing an equivalence relation $\sim$ in S as follows: $i_1 \sim i_2$ iff $\exists\, x \in S$ such that $i_1 + x = i_2 + x$, we get a new Γ-hemiring $S^* = \{x^* \mid x \in S\}$, where x^* is the equivalence class containing x, in a natural way (i.e. $i_1^* + i_2^* = (i_1 + i_2)^*$, $i_1^* \alpha i_2^* = (i_1 \alpha i_2)^*$) and a natural homomorphism η from the hemiring S to S^*. Xu called the inverse image $\eta^{-1}(J(S^*))$ the semiradical $\sigma(S)$ of S. He also proved that the right Jacobson radical and the semiradical of a Γ-hemiring coincide ([39], [40]).

Wu Pingsan and Ye Youpei studied the Jacobson radicals of near-rings. Wu proved that if I is an ideal and at the same time a direct summand of a near ring N, then $J_v(I) = J_v(N) \cap I$ for $v = 0, 1, \frac{1}{2}$ ([35]). This solves problem 7 of G. Pilz [31]. In [46] Ye proved that $J_2(\mathbf{Z}_2[x], +, \circ)$ is the additive subgroup generated by $\{x + x^a \mid a \in \mathbf{N}, 3 \nmid a\}$ and $\{x^3 + x^{3b} \mid b \in \mathbf{N}\}$, thus completing the description of $J_2(F[x], +, \circ)$ for fields F studied by Clay and Doi in [15].

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Group Graded Rings, Smash Products and Additive Categories

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Introduction For rings graded by a finite group the smash products of the rings with the grading groups play an important part in the duality theory that allows to relate properties of graded nature to ungraded properties. In this paper we propose to study similar topics but for rings graded by arbitrary groups and one of the new ingredients we introduce is the use of small additive categories associated to the smash products. First we derive the duality theorem for actions and coactions in Section 2, then we turn to study some properties of the Jacobson and the Baer radical in Section 3 and we provide some applications in Section 4. These applications center around the primitivity or simplicity of the smash product and a reinterpretation of the Wedderburn theorem for the small additive category associated to it. We have chosen a presentation that makes the material adaptable to the case of gradations by G-sets in the sense of [4], because we

aim to come back to this case in forthcoming work.

The use of smash products in graded ring-or module theory, originating from a few observations of G. Bergman, is very wide-spread at this moment, we hope to have avoided essential overlaps with the recent work of a.o. M. Cohen, S. Montgomery [1], C. Năstăsescu and Rodino [2], D. Quin [5], ..., that inspired us for this paper. We use [3] as a basic reference on graded rings.

1. Preliminaries

Let G be any group and A a G-graded ring, $A = \bigoplus_{g \in G} A_g$. The category of G-graded left A-modules will be denoted by A-gr; notation and terminology stems from [3].

Consider M and N in A-gr; we say that N weakly divides M if it is isomorphic to a graded direct summand of the direct sum $M^{(t)}$ of a finite number of copies of M. We say that M and N are weakly isomorphic in R-gr if each one weakly divides the other. A graded A-module M is said to be weakly G-invariant if M is weakly isomorphic to $M(\sigma)$ for all $\sigma \in G$, where $M(\sigma)$ is the shifted module obtained by taking the ungraded A-module M with gradation defined by $M(\sigma)_\tau = M_{\tau\sigma}$ for all $\tau \in G$. For M and N in A-gr we let $\mathrm{HOM}_A(M, N)_\rho$ be the additive group of graded homomorphisms of degree $\rho \in G$ and $\mathrm{HOM}_A(M, N) = \bigoplus_{\rho \in G} \mathrm{HOM}_A(M, N)_\rho$. The morphisms in A-gr are then given by $\mathrm{Hom}_{A\text{-}gr}(M, N) = \mathrm{HOM}_A(M, N)_e$, where e is the neutral element of G.

Recall Theorem I.5.1. of [3], p.43: for a graded A-module M, $\mathrm{END}_A(M)$ is strongly graded by G (i.e. a ring R graded by G is said to be strongly graded by G if $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$) if and only if M is weakly G-invariant. We say that $M \in A$-gr is G-invariant if for all $\sigma \in G$, $M \cong M(\sigma)$ in A-gr.

Consider the matrix ring $M_n(A)$ over the G-graded ring A. This matrix ring may be viewed as a graded endomorphism ring of a gr-free A-module if we equip it with a G-gradation depending on a set $\bar{\sigma} = \{\sigma_1, \cdots, \sigma_n\} \subset G$ as follows: $(M_n(A)(\bar{\sigma}))_\lambda = (A_{\sigma_i \lambda \sigma_j^{-1}})_{ij}$. If A is strongly

graded by G then for any choice of $\overline{\sigma}$ the ring $M_n(A)(\overline{\sigma})$ is also strongly graded by G and if A is a crossed product $A_e * G$ then for any $\overline{\sigma}$, $M_n(A)(\overline{\sigma})$ is a crossed product over $M_n(A_e)$ too.

A graded ring A is said to be gr-semisimple if A-gr a semisimple category i.e. if any $M \in A$-gr is a direct sum of gr-simple objects (being just simple objects in A-gr). Thus A is gr-semisimple if and only if $A = L_1 \oplus \cdots \oplus L_m$ for some minimal graded left ideals of A; A is gr-simple if it has such a decomposition with $\mathrm{HOM}_A(L_i, L_j) \neq 0$ for every $i, j = 1, \ldots, m$, or equivalently if $L_j \cong L_i(\sigma_{ij})$ for some $\sigma_{ij} \in G$. A gr-simple ring A is gr-uniformly simple if it has a decomposition as above with $L_i \cong L_j$ in R-gr for $i, j = 1, \ldots, m$. If A is gr-simple then A_e is only semisimple in general but if A is gr-uniformly simple then A_e is a simple ring. A gr-simple ring A having a G-invariant graded simple object is necessarily gr-uniformly simple. Generally a G-graded ring A is said to be gr-uniformly primitive if there exists a faithful G-invariant gr-simple module. In particular, if A is gr-uniformly primitive and gr-simple then it is also gr-uniformly simple. Let us recall the graded version of Wedderburn's theorem as it is given in [3]p.47. The G-graded ring A is gr-simple, resp. gr-uniformly simple, if and only if $A \cong M_n(\Delta)(\overline{\sigma})$ for some graded-division ring Δ and a $\overline{\sigma} \in G^n$, resp. $A \cong M_n(\Delta)(\overline{\sigma}) = M_n(\Delta)$ where $\overline{\sigma} = \{e, \ldots, e\} \in G^n$.

That the gr-uniformly simple rings are exactly the gr-simple rings having a G-invariant gr-simple module follows after Theorem in Section 4 (this also makes the terminology in defining gr-uniformly primitive rings consistent and adequate).

We now recall the notion of smash products. Let A be a G-graded ring with unit (as always unless otherwise mentioned). We define the smash product $A \# G$ as the free module $\oplus_{g \in G} A p_g$ with multiplication given by: $(a p_h)(b p_g) = a b_{hg^{-1}} p_g$, for $g, h \in G$ and $a, b \in A$. It is clear that $A \# G$ is a ring but without unit when G is infinite. We may rephrase the multiplication rule in $A \# G$ as follows: for $a_x \in A_x$, $b_y \in A_y$,

$(a_x p_h)(b_y p_g) = a_x b_y p_g$ if $h = yg$ and $(a_x p_h)(b_y p_g) = 0$ if $h \neq yg$. Hence in any case $(A_x p_h)(A_y p_g) \subset A_{xy} p_g$.

The smash product $A \# G$ determines a small additive category $G_{\#}$ by taking the elements of G for the objects of $C_{\#}$ and putting $\mathrm{Hom}(g, h) = A_{gh^{-1}} p_h$. Furthermore we may provide a matrix form representation for $A \# G$ by considering (possibly infinite) matrices with rows and columns indexed by G with $\mathrm{Hom}(h, g)$ in the (h, g)-position, i.e. $A \# G \cong (\mathrm{Hom}(h, g))_{(h,g)} \cong (A_{hg^{-1}} p_g)_{(h,g)} \cong (A_{hg^{-1}})_{(h,g)}$, with matrix addition and multiplication, and where every matrix is assumed to have only a finite number of nonzero entries.

We write $A \# G$-mod for the category of all $A \# G$-modules M with the property $(A \# G)M = M$ (we consider left modules unless otherwise stated), so an $M \in A \# G$-mod may be decomposed as $M = \bigoplus_{g \in G} e_{g,g} M$, where $e_{g,g} = 1_e p_g$ in the matrix presentation of $A \# G$.

1.1. Lemma The set $\{e_{g,g} = 1_e p_g, g \in G\}$ is a system of orthogonal idempotent in $A \# G$ and $A \# G = \sum_{h,g \in G} e_{h,h}(A \# G) e_{g,g}$ (this says that $A \# G$ has local identities).

Proof Straightforward, using the multiplication rules:

$$e_{h,h}(a_x p_g) = (1_e p_h)(a_x p_g) = a_x p_g \text{ if } h = xg \text{ or } 0 \text{ otherwise}$$

$$(a_x p_g) e_{h,h} = (a_x p_g)(1_e p_h) = a_x p_h \text{ if } g = h, \text{ or } 0 \text{ otherwise}$$

1.2. Corollary If $(A \# G)M^* = M^*$ then $M^* = \bigoplus_{g \in G} e_{g,g} M^*$.

To a graded A-module $M = \bigoplus_{g \in G} M_g$ we associate the $A \# G$-module $M^* = (M_e, M_h, \ldots, M_g \ldots)^t$, using the matrix representation of $A \# G$ and matrix multiplication (on the left) on M^*. Conversely, every $A \# G$-module N may be represented by $(e_{e,e}N, e_{h,h}N, \ldots, e_{g,g}N, \ldots)^t$ and we may associate to it the G-graded A-module N_* given by $N_* = \bigoplus_{g \in G} e_{g,g} N$ and scalar multiplication by (homogeneous) elements of A is the one induced in the obvious way by the scalar matrix multiplication on N. A graded morphism $\alpha: M \to M'$ in A-gr (hence of degree zero) takes M_g to M_g' for every $g \in G$ and thus we may view α as a morphism $M^* \to$

$(M')^*$ ($A\#G$-linearity is easily checked). Conversely an $A\#G$-linear β: $N \to N'$ defines a graded morphism $\beta: N_* \to N_*'$. So we obtain:

1.3. Proposition The correspondences $M \mapsto M^*$, $N \to N_*$, are inverse to each other and they define an isomorphism between A-gr and $A\#G$-mod. As a consequence, the representation type of the G-graded ring A is the same as the representation type of $A\#G$.

2. Dualities

The following lemma, pointed out to us by Luo Yun Lun, may exist in some generality in the literature, since we have no reference handy we include it here.

2.1. Lemma Let R be a ring, in general without 1, with a system of matrix units $\{e_{i,j}; i, j \in I\}$, in general with I being infinite, such that the following condition holds: $R = \sum_{i,j\in I} e_{i,i}Re_{j,j}$, then $R \cong M_{|J|}(S)$ where $S = e_{1,1}Re_{1,1}$.

Proof We define a map $\alpha: M_{|I|}(S) \to R$, $\sum s_{i,j}\bar{e}_{i,j} \mapsto \sum e_{i,1}s_{i,j}e_{1,j}$ and the reader may check all claims.

2.2. Theorem (Duality for actions) Let S be a ring with unit and G a group acting on S by automorphisms, let the action be given by the group morphism $\varphi: G \to \mathrm{Aut}\, S$. The skew group ring $S *_\varphi G$ is the free S-module $S\{\sigma, \sigma \in G\}$ with multiplication: $(s_1\sigma_1)(s_2\sigma_2) = s_1 s_2^{\sigma_1^{-1}}\sigma_1\sigma_2$ where $s_1, s_2 \in S$, $\sigma_1, \sigma_2 \in G$. Then $(S *_\varphi G)\#G \cong M_{|G|}(S)$.

Proof We aim to provide a complete system of matrix units i.e. a system of matrix units satisfying the condition of the lemma, say $\{e_{g,h}; g, h \in G\}$ such that $S = e_{e,e}((S *_\varphi G)\#G)e_{e,e} \cong S$.

Put $e_{g,k} = (1 * gh^{-1})ph \in (S *_\varphi G)\#G$.

First it is easy to calculate:

$$e_{g,h}e_{x,y} = (1 * gh^{-1}ph)\cdot(1 * xy^{-1})p_y = (1 * gh^{-1})(1 * xy^{-1})_{hy^{-1}}p_y$$

so if $h = x$ then $e_{g,h}e_{x,y} = (1 * gy^{-1})p_y = e_{g,y}$, and if $h \neq x$ then $e_{g,h}e_{x,y} = 0$. Furthermore for $s \in S$; $g, h, k \in G$ we also calculate:

$e_{g,g}(s * h)p_k = (1 * e)\ p_g(s * h)p_k = (s * h)p_k$ if $g = hk$, and

zero otherwise.

$(s * h)p_k e_{g,g} = (s * h)p_k(1 * e)p_g = (s * h)p_k$ if $k = g$, and zero otherwise.

Then $(S *_\varphi G)\# G = \sum_{g,h\in G} e_{g,g}((S *_\varphi G)\# G)e_{h,h}$ is easily verified and

$e_{e,e}((S *_\varphi G)\# G)e_{e,e} \cong S$ follows from: $e_{e,e}(s * h)e_{e,e} = s * e$ if $h = e$ or zero otherwise.

2.3. Theorem(Duality for coactions) Let A be a G-graded ring with unit. We let G act on $A\# G$ by putting: $(\sum_{h,k\in G} a_h p_k)^g = \sum_{h,k\in G} a_h p_{kg}$ (this is indeed an action by automorphisms) and then we form the skew ring $(A\# G) * G$. We obtain: $(A\# G) * G \simeq M_{|G|}(A)$.

Proof Multiplication in $(A\# G) * G$ is given by the rule:

$$\begin{aligned}((ap_g) * h)((bp_x) * y) &= (ap_g)(bp_x)^{h^{-1}} * hy = (ap_g)(bp_{xh^{-1}}) * hy \\ &= (ab_{g(xh^{-1})^{-1}})p_{xh^{-1}} * hy \\ &= (ab_{ghx^{-1}})p_{xh^{-1}} * hy\end{aligned}$$

Put $e_{g,gh} = (1.p_g) * h$.

Now $e_{g,gh}e_{x,xy} = ((1p_g) * h)((1p_x) * y) = (1p_{xh^{-1}}) * hy = e_{xh^{-1},xy}$ if $gh = x$ (then it equals $e_{g,xy}$ too) but it is zero otherwise, hence $\{e_{g,gh}; g, h \in G\}$ is a system of matrix units.

Next one calculates in a straightforward way:

$e_{x,x}((a_k p_h) * g) = ((1.p_x) * e)((a_k p_h) * g) = (a_k p_h) * g$

if $x = kh$ and zero if $x \neq kh$;

$((a_k p_h) * g)e_{x,x} = ((a_k p_h) * g)((1p_{hg}) * e) = (a_k p_h) * g$ if $x = hg$ and zero if $x \neq hg$.

From these equalities one easily obtains the fact that $\{e_{g,gh}; g, h \in G\}$ is a complete system. Furthermore we may calculate $e_{e,e}((A\# G) * G)e_{e,e} = \sum_{g\in G}(A_g p_{g^{-1}}) * g = B$. If we define $\alpha: B \to A$, $(r_g p_g^{-1}) * g \mapsto r_g$ then α defines an isomorphism and then the claim has been established. Let us just check that α is a ring morphism:

$$\alpha[((r_g p_g^{-1}) * g)((r_h p_{h^{-1}}) * h)] = \alpha[(r_g r_h p_{h^{-1}g^{-1}}) * gh]$$

$$= r_g r_h = \alpha((r_g p_{g^{-1}}) * g)\alpha((r_h p_{h^{-1}}) * h)$$

3. Radicals of Smash Products

Let A be a G-graded ring with unit. If $I = I_e \oplus I_h \oplus \cdots \oplus I_g \oplus \cdots$ is a graded subring of A then $I \# G$ is a subring of $A \# G$; if I is a graded ideal then $I \# G$ is an ideal of $A \# G$.

3.1. Lemma Let M be a graded A-module, then we have:

$$(\mathrm{Ann}_A(M)) \# G = \mathrm{Ann}_{A\#G}(M^* \oplus M(h)^* \oplus \cdots \oplus M(g)^* \oplus \cdots) = \bigcap_{x \in G} \mathrm{Ann}_{A\#G}(M(x)^*)$$

Proof In matrix presentation the annihilation condition is given as

$$\begin{pmatrix} a_e & a_{h^{-1}} & \cdots & a_{g^{-1}} \\ a_h & a_e & \cdots & a_{hg^{-1}} \\ \cdots & \cdots & \cdots & \cdots \\ a_g & a_{gh^{-1}} & \cdots & a_e \end{pmatrix} \begin{pmatrix} M_x \\ M_{hx} \\ \cdots \\ M_{gx} \end{pmatrix} = 0$$

Let $\mathrm{Ann}_{A\#G}(M(x)) = (I^{(h)}_{hg^{-1}}(x))_{(h,g)}$, then $I^{(e)}_e(x) = \mathrm{Ann}_{A_e}(M_x)$, $I^{(e)}_{h^{-1}}(x) = \mathrm{Ann}_{A_{h^{-1}}}(M_{hx}), \cdots, I^{(h)}_e(x) = \mathrm{Ann}_{A_e} M_{hx}$.

Thus $\bigcap_{x\in G}\mathrm{Ann}_{A\#G}M(x) = (I^{h}_{gh^{-1}})_{(h,g)}$, where: $I^{(e)}_e = \bigcap_{x\in G}\mathrm{Ann}_{A_e}(M_x) = \mathrm{Ann}_{A_e}(M)$, $I^{(h)}_h = \bigcap_{x\in G}\mathrm{Ann}_{A_h}(M_x) = \mathrm{Ann}_{A_h}(M)$, $I^{(h)}_e = \bigcap_{x\in G}\mathrm{Ann}_{A_e}(M_{hx}) = \bigcap_{y\in G}\mathrm{Ann}_{A_e}(M_y) = \mathrm{Ann}_{A_e}(M)$. This learns that

$$\bigcap_{x\in G}\mathrm{Ann}_{A\#G}(M(x)) = (\mathrm{Ann}_{A_{hg^{-1}}}(M))_{h,g} = \mathrm{Ann}_A(M)\# G$$

3.2. Corollary For $N \in A\#G$-mod we have:

$$\bigcap_{x\in G}\mathrm{Ann}_{A\#G}((N_*(x))^*) = \mathrm{Ann}_A(N_*)\# G$$

3.3. Lemma For $M \in A$-gr we have the following equivalences:

1. M is gr-simple if and only if $M(x)$ is gr-simple, for all $x \in G$.
2. M is gr-simple if and only if M^* is a simple $A\#G$-module.

Proof 1 It is easy **2**. follows from the isomorphism given in Proposition 1.3.

As a straightforward consequence of the foregoing corollary and lemma we obtain:

3.4. Theorem $J(A\#G) = J^g(A)\#G$, where $J(-)$ denotes the Jacobson radical and $J^g(-)$ the graded Jacobson radical. We devote the sequel of this section to proving the equivalent of Theorem 3.4. for the Baer-radical of modules.

An $M \in A$-gr is called a gr-prime A-module if for every nonzero graded submodule N of M and every graded ideal I of A, $IN = 0$ implies $I \subset \mathrm{Ann}_A(M)$. Note that $\mathrm{Ann}_A(M)$ is graded. For a ring B, possibly without unit, an $N \in B$-mod is prime if for every B-submodule $P \neq 0$ and ideal J of B such that $JP = 0, J \subset \mathrm{Ann}_B(P)$ holds. Note that in case B has no unit, modules X are always assumed to satisfy $BX = X$. The Baer radical of B, denoted by $\mathbf{B}(B)$, is the intersection of all $\mathrm{Ann}_B(N)$ for prime B-modules N. The graded Baer radical of a G-graded ring A is the intersection of all annihilators (and these are graded) of gr-prime A-modules; we denote this radical by $\mathbf{B}^g(A)$. Clearly, if $M \in A$-gr is prime then each $M(x)$ is also gr-prime. We first proceed to prove that $M^* \oplus M(h)^* \oplus \cdots \oplus M(g)^* \oplus \cdots$ is then a prime $A\#G$-module. Observation: a graded submodule N of a gr-prime M is also gr-prime.

3.5. Lemma Let $\{M_i, i \in \mathrm{I}\}$ be a family of gr-prime A-modules having the same annihilator I in A, then $\bigoplus_{i\in \mathrm{I}} M_i$ is also gr-prime with annihilator I.

Proof Put $P = \bigoplus_{i\in \mathrm{I}} M_i$, then $P_g = \bigoplus_{i\in \mathrm{I}}(M_i)_g$ for all $g \in G$. Let J be a graded ideal of A and N a nonzero graded A-submodule of P such that $JN = 0$. We aim to prove $J \subset \mathrm{Ann}_A P = \bigcap_{i\in J}\mathrm{Ann}_A M_i = I$. The projection $P \rightarrow M_i$ is denoted by π_i; put $N_i = \pi_i N$, then $0 \neq N \subset \bigoplus_{i\in J} N_i$. Obviously, $JN = 0$ implies $JN_i = 0$ for all $i \in \mathrm{I}$. Since at least one N_i is nonzero and M_i is a gr-prime module we do obtain $J \subset \mathrm{Ann}_A(M_i) = \mathrm{Ann}_A(P) = I$.

3.6. Proposition If M is a gr-prime module then $(\bigoplus_{x\in G} M(x))^* = P^*$ is a prime $A\#G$-module.

Proof Put $\mathrm{Ann}_A(M) = \bigoplus_{h\in G} I_h$. By the above lemma and remarks:

$$\mathrm{Ann}_{A\#G}(P^*) = \mathrm{Ann}_{A\#G}\left(\begin{pmatrix} M_e \\ M_h \\ \vdots \\ M_g \\ \vdots \end{pmatrix} \oplus \begin{pmatrix} M_h \\ \vdots \\ M_{g^h} \\ \vdots \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} M_g \\ M_{hg} \\ \vdots \\ M_{g^2} \\ \vdots \end{pmatrix} \oplus \cdots\right)$$

$$= \mathrm{Ann}_{A\#G}(M, \ldots, M)^t = \mathrm{Ann}_A(M)\#G$$

Consider a nonzero $A\#G$-submodule $N = (N(e), N(h), \ldots, N(g), \ldots)^t$ of P^*. Put $\mathrm{Ann}_{A\#G}(N) = J = (J^{(h)}_{hg^{-1}})_{(h,g)}$; we aim to establish $J \subset I\#G$.

We say that an $A\#G$-submodule T of P^* is a homogeneous submodule if T may be written as $(T^{(e)}, T^{(h)}, \cdots, T^{(g)}, \cdots)^t$ where each $T^{(g)}$ is a graded A-submodule of M, for $g \in G$. Put $(T^{(g)})_h = T^{(g)}_{gh}$. Let $\overline{N}$ be the unique minimal homogeneous $A\#G$-submodule containing N. One easily checks $\mathrm{Ann}_{A\#G}(\overline{N}) = \mathrm{Ann}_{A\#G}(N)$ and therefore $J.\overline{N} = 0$. So we obtain:

$0 = J.\overline{N}$

$$= \begin{pmatrix} J^{(e)}_e & J^{(e)}_{h^{-1}} & \cdots \\ J^{(h)}_h & J^{(h)}_e & \cdots \\ \vdots & \vdots & \end{pmatrix} \begin{pmatrix} N^{(e)}_e & + & N^{(e)}_h & + & \cdots & + & N^{(e)}_g & + & \cdots \\ N^{(h)}_h & + & N^{(h)}_{hh} & + & \cdots & + & N^{(h)}_{h_g} & + & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & + & 0 & + & \cdots & + & 0 & & \\ 0 & + & 0 & + & \cdots & + & 0 & & \end{pmatrix}$$

Some rows may be zero, but we can find a nonzero column, say the first one. Since $N^{(e)}_e \oplus N^{(h)}_h \oplus \cdots \oplus N^{(g)}_g \oplus \cdots$ is a nonzero graded A-module and $\sum_{h\in G} J^{(e)}_{h-1}$ is a right ideal of A, the annihilation condition leads to $\left(\sum_{h\in G} J^{(e)}_{h^{-1}}\right)\left(\sum_{h\in G} N^{(h)}_h\right) = 0$, hence $\sum_{h\in G} J^{(e)}_{h^{-1}} \subset \mathrm{Ann}_A M$ by the gr-prime condition on M. Applying a similar argument to the row $(J^{(h)}_h, J^{(h)}_e, \cdots, J^{(h)}_{hg^{-1}}, \cdots)$ we obtain that $\sum_{h\in G} J^{(h)}_{hg^{-1}} \subset \mathrm{Ann}_A M$ and therefore we arrive at $J \subset \mathrm{Ann}_A(M)\#G = \mathrm{Ann}_{A\#G}(P^*)$.

3.7. Theorem $\mathbf{B}(A\#G) = \mathbf{B}^g(A)\#G$.

Proof It will suffice to establish that a prime $A\#G$-module N

corresponds to a gr-prime A-module N_*. Therefore consider a nonzero graded submodule P of N_* and let $IP = 0$ for some graded ideal I of A, $I = I_e + I_h + \cdots$. Clearly we have that P^* is an $A \# G$-submodule of N and $I \# G$ is an ideal of $A \# G$ such that $(I \# G)P^* = 0$. Since N is prime we obtain $(I \# G)N = 0$ hence $IN_* = 0$ or $I \subset \mathrm{Ann}_A(N_*)$.

4. Some Applications

We first use the connection between the graded ring and the small additive category associated to the smash product in order to derive some results of graded nature from the structure theorems for additive categories appearing in [6]. We recall:

4.1. Theorem Let C be a small additive category with object set Σ *and write* ${}_\alpha C_\beta$ for $\mathrm{Hom}_C(\alpha, \beta)$. We may form the generalized matrix ring $({}_\beta C_\gamma)_{(\beta,\gamma)}$. If H is any finite (consequently, any subset will do) subset of Σ *then we have*: $\{({}_\alpha C_\beta); \alpha, \beta \in H\} \cap J\{({}_\alpha C_\beta\}; \alpha, \beta \in \Sigma\} = J\{({}_\alpha C_\beta); \alpha, \beta \in H\}$, i.e. if $J\{({}_\alpha C_\beta); \alpha, \beta \in \Sigma\} = \{({}_\alpha J_\beta); \alpha, \beta \in \Sigma\}$ then $J\{({}_\alpha C_\beta); \alpha, \beta \in H\} = \{{}_\alpha J_\beta); \alpha, \beta \in H\}$.

4.2. Theorem If A is a G-graded ring (with unit) and H is a subgroup of G then $J^g(A^H) = J^g(A) \cap A^H = (J^g(A))^H$, where $A^H = \bigoplus_{h \in H} A_h$.

Proof By the duality theorem we have $J^g(A^H) \# H = J(A^H \# H)$. By the above theorem: $J(A^H \# H) = J(A \# G) \cap (A^H \# H) = (J^g(A) \# G) \cap (A^H \# H) = (J^g(A))^H \# H$ and form $J^g(A^H) \# H = (J^g(A))^H \# H$ the equality $(J^g(A))^H = J^g(A^H)$ follows.

Recall from [6] again:

4.3. Theorem Consider the generalized matrix ring $({}_\alpha C_\beta)_{(\alpha,\beta)}$, for the small additive category C. The category C is primitive if and only if C is a dense subcategory of VD, the category of D-vector spaces over a division ring D ("dense" means that ${}_\alpha C_\beta$ is a dense subset of $\mathrm{Hom}_D(\alpha, \beta)$, where α, β are D-vectorspaces up to identification of C in VD, i.e. for any $n \in \mathbf{N}$ and D-independent elements $a_1, \cdots, a_n$ of α, any elements $b_1, \cdots, b_n$ of β, there exists $\theta \in {}_\alpha C_\beta$ such that $a_i\theta = b_i$, $c = 1, \cdots, n$.

From [6] we retain the following.

4.4. Theorem Let C be a simple additive category and for all $\alpha \in \Sigma$, ${}_\alpha C_\alpha$ is Artinian then C is isomorphic to a full subcategory of the category FVD of all finite dimensional D-vector spaces over the division ring D.

Remark A C-module M may be written as the column $(M_\alpha, \alpha \in \Sigma)^t$ with matrix action of $({}_\alpha C_\beta)_{(\alpha,\beta)}$ defined on it. The functor $C \to VD$ takes α to $(M_\alpha)_D$ where $D = \mathrm{End}_\alpha C_\alpha(M_\alpha)$ and ${}_\alpha a_\beta \in {}_\alpha C_\beta$ to $t_{{}_\alpha a_\beta}: M_\alpha \to M_\beta$, $m \mapsto {}_\alpha a_\beta \cdot m$.

Before deriving the corresponding results on graded rings we need the following easy lemma.

4.5. Lemma If the G-graded ring A is a direct sum of minimal graded left ideals then $A\# G$ is a direct sum of minimal left ideals.

Proof Put $A = L_1 \oplus \cdots \oplus L_j \oplus \cdots$, $L_i = (L_i)_e \oplus (L_i)_h \oplus \cdots \oplus (L_i)_g \oplus \cdots$. Then $A\# G \cong (Ahg^{-1})_{(h,g)} = \left(\sum_i (L_i)_{hg^{-1}}\right)_{(h,g)}$, and the latter clearly decomposes as a sum of minimal left ideals of $A\# G$, each one of the form

$$\begin{pmatrix} 0 & \cdots & 0 & \cdots & (L_i)_{g^{-1}} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & & (L_i)_{hg^{-1}} & \vdots & & \vdots \\ \vdots & & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

4.6. Theorem For a G-graded ring A with unit the following assertions are equivalent:

1. A is gr-uniformly primitive with all $A_h \neq 0$.

2. $A\# G$ is a primitive ring.

3. There exists a division ring D and a D-vectorspace M such that for all $h \in G$, A_h is a dense subring of $\mathrm{End}_D(M)$.

Proof 1.⇒2. Let $M = M_e + M_h + \cdots$ be a faithful G-invariant graded A-module and let $\alpha_g: M \to M(h)$ be fixed isomorphisms for $h \in G$. Put $N = \{(m_e^{(e)} + \alpha_h m_e^{(e)} + \cdots + \alpha_g m_e^{(e)}, \cdots, \alpha_h m_e^{(h)} + m_e^{(h)} + \cdots +$

$\alpha_{hg^{-1}} m_e^{(e)} + \cdots, \cdots)^t; m_e^{(e)}, m_e^{(h)}, \cdots \in M_e\}$. It is clear that N is an $A \# G$-submodule of $M^* + M(h^{-1})^* + \cdots + M(g^{-1})^* + \cdots$. Now if $I_h \subset A_h$ is such that $I_h(m_e^{(g)} + \alpha_h m_e^{(g)} + \cdots) = 0$ then $I_h m_e^{(g)} = I_h \alpha_h m_e^{(g)} = \cdots = 0$, and from this we deduce that: $\mathrm{Ann}_{A\#G}(N) = (\mathrm{Ann}_A(M)) \# G = 0$, because the first term equals $\bigcap_{h\in G} \mathrm{Ann}_{A\#G} M(h)^*$. Consequently N is a faithful $A \# G$-module and since $N \cong M^*$ it is also a $A \# G$-simple module (apply lemma 3.3.(2)) and therefore $A \# G$ is a primitive ring.

2.⇒3. The primitive additive category $(\overset{(h)}{A}_{hg^{-1}})_{(h,g)}$ with object set G is isomorphic to a dense additive category, over some division ring D, in view of Theorem 4.3. Paying attention to, and using the notation of, the remark preceding Lemma 4.5. we see that $(M_e)_D \cong (M_h)_D \cong \ldots \cong (M_g)_D \cong \ldots$ and we also have embeddings:

$$A_{hg^{-1}} p_h = \overset{(h)}{A}_{hg^{-1}} \hookleftarrow \mathrm{Hom}_D(M_h, M_g) = \mathrm{Hom}_D(M_e, M_e)$$

$$A_{(hx)(gx)^{-1}} p_{hx} = \overset{(hx)}{A}_{hg^{-1}} \hookleftarrow \mathrm{Hom}_D(M_{hx}, M_{gx}) = \mathrm{Hom}_D(M_e, M_e)$$

and $\overset{(h)}{A}_{hg^{-1}} = \overset{(hx)}{A}_{hg^{-1}}$ as a subring of $\mathrm{Hom}_D(M_e, M_e)$. It follows that $A_e \cong A_e^{(e)} = A_e^{(h)} = \cdots$ is a dense subring of $\mathrm{Hom}_D(M_e, M_e)$ and the same holds for $A_{hg^{-1}} \cong \overset{(e)}{A}_{hg^{-1}} = \overset{(h)}{A}_{hg^{-1}} = \cdots$.

3.⇒1. Suppose $A = \bigoplus_{h\in G} A_h$ and all A_h being dense subrings of $\mathrm{End}_D(V)$. Now we define V_g equal to V as a right D-vector space but defining $a_h \cdot n_g = a_h(n_g)$ where on the right we view a_h as the linear transformation $V \to V$ but putting $a_h(n_g)$ in V_{hg}. Put $M = \bigoplus_{g\in G} V_g$. Clearly M is a graded A-module and it is G invariant as well as faithfull.

Remark In the embedding of $\overset{(h)}{A}_{hg^{-1}}$ in $\mathrm{Hom}_D(M_e, M_e)$ one has to be careful how to write the order of multiplication in $\mathrm{Hom}_D(M_e, M_e)$ (or else write the opposite ring).

Considering gr-simple modules we may also obtain the following extension of the graded version of the Wedderburn theorem as given in [3], in particular in the gr-uniformly simple case.

4.7. Theorem Let A be a G-graded ring such that A_e is left

Artinian, then the following statements are equivalent:

1. A is gr-quasi-simple(no graded ideals that are nontrivial)and it has a G-invariant gr-simple module.

2. $A \# G$ is simple.

3. For all $h \in G$, $A_h = M_n(D)$ for some n not depending on h and some division ring D.

4. A is gr-uniformly simple with all $A_h \neq 0$.

Proof 1.⇒3. The foregoing theorem implies that A is gr-uniformly primitive and each A_h is dense in $\mathrm{End}_D(V)$ for some D-vectorspace V. But A_e is then a left Artinian dense subring of $\mathrm{End}_D(V)$and therefore V is finite dimensional over D; that all A_h are now isomorphic to $M_n(D)$is obvious.

3.⇒2. Trivial

2.⇒3. Similar argumentation as in foregoing theorem but using Theorem 4.4. instead of Theorem 4.3. Since 4.⇒1. is trivial, the proof is finished if we establish 3.⇒4.

3.⇒4. Let C_i be the i-th column of $M_n(D)$; put $C_{i,h}$ equal to this column viewed as a left module in $A_h \cong M_n(D)$and define $L_i = \bigoplus_{h\in G} C_{i,h}$. It is clear that L_i is a minimal graded left ideal of A and that $A = L_1 \oplus \cdots \oplus L_n$. Moreover $L_i \cong L_j$(for all i, j)as graded A-modules. That each L_i is G-invariant is obvious from the definition, so the claim follows.

4.8. Corollary 1 The conditions in the theorem are equivalent to the following condition.

1. A is gr-simple having a unique, up to isomorphism, gr-simple A-module(this is then G-invariant too).

2. The Wedderburn theorem on p.47 of [3]proves that a gr-uniformly simple ring A is of the form $M_n(\Delta)(\bar{e})$i.e. $(M_n(\Delta))_\lambda \cong M_n(\Delta_\lambda) \cong M_n(\Delta_e) = M_n(D)$, $D = \Delta_e$ a division ring. From the above it follows thus that a gr-uniformly simple ring A with $A_h \neq 0$ for all $h \in G$, has a G-invariant(and up to isomorphism unique)gr-simple module; the converse

was mentioned in [3] but this implication was missing there.

4.9. Note If for every $h \in G$ such that $A_h \neq 0$ we have $\Delta_h \neq 0$ then also $A \cong M_n(\Delta)(\bar{e})$ and it is strongly graded by the subgroup sup(Δ) of G(those $g \in G$ such that $\Delta g \neq 0$). However, if $G \neq \mathrm{Sup}(\Delta)$ then A is not gr-uniformly simple. As an example consider the **Z**-graded ring $A = M_2(k[X, X^{-1}])(0, 1)$ where $\deg X = 2$. One easily verifies that this ring is even strongly graded by **Z** but its part of degree zero equals $k \oplus k$ and as this is not simple A cannot be gr-uniformly simple!

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Structure of Graded Primitive Rings*

Keywords gr-primitive ring, gr-dense ring of linear transformations.

Let G be an arbitrary group, finite or infinite, and A be a G-graded ring. i. e. A is an associative ring and $A = \bigoplus_{g \in G} A_g$ (direct sum of additive subgroups A_g) with property: $A_g \cdot A_h \subseteq A_{gh}$, $g, h \in G$. Let M be a graded A-module, i. e. M is right A-module and $M = \bigoplus_{g \in G} M_g$ with property: $M_g \cdot A_h \subseteq M_{gh}$, $g, h \in G$. Fundamental definitions concerning graded rings and graded modules can be found, for example, in [1] and [2].

A graded ring A is said to be gr-primitive, if there exists a faithful gr-simple A-module. We have proved structure theorems of Artin graded rings[2] and of gr-uniformly primitive rings[3]. In this note the structure of arbitrary gr-primitive rings is given.

First of all, we give a general method to construct G-gr-primitive rings.

Let K be an ordinary (ungraded) division ring. Take a left K-space M_g for every element g of group G and form their direct sum

$$M = \bigoplus_{g \in G} M_g \qquad (*)$$

Suppose that $E_g = \{\varphi \in \mathrm{End}_K M \mid M_h \varphi \subseteq M_{hg}, \forall h \in G\}$ and let

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$$E = \sum_g E_g = \bigoplus_g E_g \subseteq \mathrm{End}_K M.$$

It is easy to see that E is a G-graded ring with identity 1.

Definition 1 The G-graded ring E shown above is called G-graded ring of linear transformations on K-space M(with respect to the decomposition(*)). A graded subring $A = \bigoplus_g A_g$(with or without 1)of E is said to be gr-dense ring of linear transformations, if for any given finite K-independent set$\{x_1, \cdots, x_n\} \subseteq M_g$, any $h \in G$ and any given finite set$\{y_1, \cdots, y_n\} \subseteq M_{gh}$, where n is an arbitrary positive integer, there exists an element $a \in A$ such that $x_i a = y_i$, $\forall\, i$, it means that if we restrict actions of elements of A_h on M_g, then A_h is a dense subset of $\mathrm{Hom}_k(M_g, M_{gh})$.

It is easy to prove that every gr-dense ring of linear transformations on a K-space M(respectively to an arbitrary decomposition(*))is G-gr-primitive and that M itself is its faithful gr-simple module. Now we are going to prove that the inverse is also true and therefore all G-gr-primitive rings can be obtained in this way.

We firstly discuss structure of gr-primitive ring by the method used in ordinary ring theory [4]. Here we shall give statements only and omit their proofs. Then, improving those results we can get what we want.

Definition 2 Let D be a G-graded division ring. (i.e. every nonzero homogeneous element of D has an inverse element)and let M be a left G-graded D-module. A finite set of homogeneous elements $m_1, \cdots, m_n$ in M is said to be D-independent, if $\sum_i d_i m_i = 0$, implies $d_i = 0$, $\forall\, i$, where d_i are homogeneous elements of D. A set of homogeneous elements of M is said to be D-independent, if any of its finite subset is D-independent. A set N is said to be a basis of M, if N is D-independent and is a generating set of M.

It is easy to see that a set$\{m_\alpha, \alpha \in I\}$of homogeneous elements of M is D-independent iff the set$\{d_\alpha m_\alpha, \alpha \in I\}$is D-independent, where d_α are nonzero homogeneous elements of D.

Proposition 1 Any graded D-module over a graded division ring D has a basis.

Definition 3 Let D be a graded division ring and $M = \bigoplus_g M_g$ be a graded D-module. $\mathrm{End}_D(M)$, as usual, denotes the endomorphism ring of ordinary(ungraded) D-module M. For $g \in G$, let $T_g = \{\varphi \in \mathrm{End}_D(M) \mid M_h\varphi \subseteq M_{hg}, \forall h \in G\}$, and $\mathrm{END}_D M = \sum_g T_g = \bigoplus_g T_g \subseteq \mathrm{End}_D(M)$. $\mathrm{END}_D M$ is G-graded ring(with identity 1)and is said to be graded endomorphism ring of graded D-module M. A graded subring $A = \bigoplus_g A_g$ (with or without 1)of $\mathrm{END}_D M$ is said to be gr-dense ring(over graded D-module M), if for any given finite D-independent set $\{m_1, \cdots, m_n\} \subseteq M$ and any given finite set $\{y_1, \cdots, y_n\} \subseteq M$, where n is an arbitrary positive integer, there exists $a \in A$ such that $m_i a = y_i$, $\forall i$.

It is easy to observe that gr-dense rings are gr-primitive rings.

Schur Lemma(Graded Version) Let A be a G-gr-primitive ring and M be a faithful gr-simple A-module. Then $\mathrm{END}_D M$(cf. Defintion 4) is a G-graded division ring.

Jacobson-Chevalley Dense Theorem(Graded Version) Let A be a G-gr-primitive ring and M be a faithful gr-simple A-module. Let $D = \mathrm{END}_A M$. Then M may be naturally considered as D-A-bimodule. By the faithfulness of M, the ring A can be naturally looked as a graded subring of $\mathrm{END}_A M$ and A must be a gr-dense ring.

Its proof is similar to that in the ungraded case(cf. [4])and therefore omitted. From this we easily get the Wedderburn-Artin theorem(graded version)[1]. We also omit its proof.

Now we prove the following.

Propositon 2 Let A be G-graded ring(with or without 1)and $M = \bigoplus_g M_g$ be gr-simple A-module. Let $D = \mathrm{END}_A M = \bigoplus_g D_g$. If $M_g \neq 0$, $g \in G$, then $\mathrm{End}_{A_e}(M_g) \simeq D_e$, where e is the identity of G.

Proof Take arbitrarily $0 \neq m_g \in M_g$. By the simplicity of M, we have

$$M = m_g A_e \oplus (\bigoplus_{e \neq h \in G} m_g A_h),$$

and therefore $M_g = m_g A_e$. For each $\varphi \in \mathrm{End}_{A_e}(M_g)$ we define a mapping φ' as follows:

$$\varphi' : M \to M$$

$$m_g a_x \to \varphi(m_g) a_x, a_x \in A_x, x \in G.$$

To show that φ' is well-defined, it is enough to prove that if $m_g a_x = 0$, then $\varphi(m_g) a_x = 0$. This is true in the case $x = e$, because φ is an endomorphism of A-module M. For arbitrary $x \in G$, $m_g a_x = 0$ implies $m_g a_x A_x^{-1} = 0$. From $a_x A_x^{-1} \subseteq A_e$ we get $\varphi(m_g) a_x A_x^{-1} = 0$ by the just mentioned fact. Suppose that $\varphi(m_g) a_x \neq 0$. Then by the simplicity of M we have $M = \varphi(m_g) a_x A$, consequently, $M_g = \varphi(m_g) a_x A_x^{-1} = 0$, which contradicts with the condition $M_g \neq 0$, and thus φ' is well-defined. It is easily verified that φ' is gr-endomorphism of degree e(i.e. φ' is an endomorphism of A-module and $\varphi'(M_h) \subseteq M_h$, $\forall h \in G$), therefore, $\varphi' \in D_e$. So, every element of $\mathrm{End}_{A_e}(M_g)$ can be extended to be an element in D_e.

Define a mapping

$$\alpha : D_e \to \mathrm{End}_{A_e}(M_g)$$

$$\varphi' \to \varphi' |_{M_g} \text{(the restriction of } \varphi' \text{ on } M_g).$$

From the above discussion we see that α is surjective. On the other hand, by the simplicity of M every endomorphism of A-module M is uniquely determined by the image of any nonzero element of M, therefore any two elements of D_e should be equal to each other, if their restrictions on M_g are the same. This means that α is injective and perserves operations. Our proposition is proved.

Proposition 3 (i)Let K be an arbitrary division ring and $A = \bigoplus_g A_g$ be a gr-dense ring of linear transformations over a left K-space M with a decomposition $M = \bigoplus_g M_g$(cf. Definition 1). Then A can be interpreted as a gr-dense ring over the right graded D-module $M = \bigoplus_g M_g$(cf. Definition 3), where D is a graded division ring and $D_e \simeq K$. (ii) Let $A = \bigoplus_g A_g$ be a

gr-dense ring over a graded D-module $M = \bigoplus_g M_g$, where D is a graded division ring. Then $A = \bigoplus_g A_g$ may be interpreted as a gr-dense ring of linear transformations over the D_e-space M with D_e-space decomposition $M = \bigoplus_g M_g$.

Proof (i) It is easy to see that $M = \bigoplus_g M_g$ may be naturally considered as a right graded A-module, and M is then a faithful gr-simple A-module. Let $D = \text{END}_A M = \bigoplus_g D_g$, where D is a graded division ring by Schur Lemma. Then, by JC dense theorem(graded verison) A is a gr-dense ring over a left graded D-module M. So, the only thing we have to prove is that $D_e \simeq K$. There should be $M_g \neq 0$ for some $g \in G$, because $M \neq 0$. By Proposition 2 we have $D_e \simeq \text{End}_{A_e} M_g$. On the other hand, from Definition 1 we see that $A_e \mid_{M_g}$ (restriction of action of A_e on M) is a dense subring in $\text{End}_K(M_g)$. Therefore, by a well-known result(cf. [4, p.32, Theorem 1]) we have $\text{End}_{A_e} M_g = K$. (i) is proved.

(ii) What we have to show is that for any finite subset $N = \{x_1, \cdots, x_n\} \subseteq M_g$, if N is D_e-independent(in D_e-space M), then it is also D-independent(in graded D-module M). This is obviously true. Supposing that $\sum d_i x_i = 0$, where $d_i \in D$ are homogeneous elements and not all zeros, without loss of generality we may assume that $d_i \in D_h$, $\forall\, i$ for some $h \in G$. Consequently $D_h \neq 0$ and then also $D_{h^{-1}} \neq 0$. Taking any $0 \neq d \in D_{h^{-1}}$, we have $0 = d \cdot \sum d_i x_i = \sum (dd_i) x_i$, where $dd_i \in D_e$, $\forall\, i$. Since N is D_e-independent, so $dd_i = 0$ and consequently $d_i = 0$, $\forall\, i$. This contradicts our assumption. (ii) is proved.

From all the above we get the main result of this note.

Theorem Let G be an arbitrary group. A G-graded ring A is gr-primitive iff A is a gr-dense ring of linear transformations over a left K-space $M = \bigoplus_g M_g$ (Definition 1), where K is a division ring.

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分次除环和 Jacobson 稠密性定理

Graded Division Rings and the Jacobson Density Theorem*

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Abstract The graded division rings and a graded version of the Jacobson density theorem are studied. As a corollary, it is shown that a right gr-Artinian ring with zero graded Jacobson radical, is left gr-Artinian.

Keywords primitive graded ring; density theorem; left gr-Artinian ring.

0 Preliminaries

In this note, we study graded division rings and gr-primitive graded rings, i.e., graded rings with a faithful gr-simple graded module. In the first section, we show that the idea of dimension of a module over a graded division ring makes sense. The second section is concerned with gr-primitive rings and a version of the Jacobson density theorem for graded rings.

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Throughout, G will denote a group with identity e, and A a G-graded ring, not necessarily with 1. Modules will be right modules unless otherwise specified. $h(A)$ will denote the homogeneous elements of A. If M and N are graded right A-modules, $\mathrm{HOM}_A(M, N) = \bigoplus_{g \in G} \mathrm{HOM}_A(M, N)_g$ where $\mathrm{HOM}_A(M, N)_g = \{\phi \in \mathrm{Hom}_A(M, N) : \phi(M_h) \subseteq N_{gh} \text{ for all } h \in G\}$[1]. Recall that if G is finite or if M is finitely generated, then $\mathrm{HOM}_A(M, N) = \mathrm{Hom}_A(M, N)$ (see[1]). $\mathrm{END}_A(M)$ is a G-graded ring.

A G-graded ring D with 1 is called a graded division ring if every element of $h(D)$ is a unit. For example, if A is a graded ring and M a gr-simple graded A-module(i.e. $MA = M$ and M has no proper graded submodules) then $D = \mathrm{END}_A(M)$ is a graded division ring, and M is a graded left D-module.

If D is a G-graded division ring, the subset $H = \{h \in G : D_h \neq 0\}$ of G is a subgroup of G. If M is a left D-module, and $M_g \neq 0$, then $D_h M_g = M_{hg}$ for all $h \in H$.

1 Modules over Graded Division Rings

It is straightforward to see that every graded module over a graded division ring is gr-free, i.e. has a basis of homogeneous elements. For, note that if a set $\{m_1, \cdots, m_n\} \subseteq h(M)$ is not D-independent then there exist $d_i \in h(D)$, with the d_i not all 0, and $\sum d_i m_i = 0$. In this section we show that every homogeneous basis has the same cardinality, and thus the notion of dimension of a graded module over a graded division ring makes sense.

Throughout, D will denote a G-graded division ring, and H will be the subgroup $\{h \in G : D_h \neq 0\}$.

A set $\{m_1, \cdots, m_n\} \subseteq M_g$ is D-independent if and only if the set is D_e-independent. For if $\sum d_i m_i = 0$, then we may assume all $d_i \in D_h$ for some $h \in G$. If $d_1 \neq 0$, then $\sum d_1^{-1} d_i m_i = 0$ with $d_1^{-1} d_i \in D_e$.

Definition 1.1 We call a G-graded left D-module M D-transitive if, for all $M_g \neq 0$, $M = DM_g = \bigoplus_{h \in H} D_h M_g$. For example, D itself is a D-transitive module.

First we see that every homogeneous basis of a D-transitive D-module has the same cardinality.

Proposition 1.2 For M a graded transitive left D-module with M_g nonzero, then any basis of the D_e-module M_g is also a homogeneous basis of M. Furthermore, all homogeneous bases of M have the same cardinal number.

Proof Suppose $\{m_i : i \in I\}$ is a D_e-basis for $M_g \neq 0$. Then, as noted above, $\{m_i : i \in I\}$ is D-independent. Furthermore, $\sum_{i \in I} Dm_i = \sum_{i \in I} DD_e m_i = D \sum_{i \in I} D_e m_i = DM_g = M$.

Now suppose $\{m_i : i \in I\} \subset h(M)$, $m_i \in M_{g_i}$, is a homogeneous basis for M. Then $M = \sum_i Dm_i$ and suppose $0 \neq M_g = \sum_i D_{gg_i^{-1}} m_i$. Then, since M is transitive, $gg_i^{-1} \in H$, i.e. $D_{gg_i^{-1}} \neq 0$, for all i. Choose $0 \neq d_i \in D_{gg_i^{-1}}$, and $M_g = \sum_i D_e d_i m_i$. Also it is clear that $\{d_i m_i : i \in I\}$ is D_e-independent. Therefore all D-bases of M have the same cardinal number.

Every D-module is D-transitive if and only if D is strongly graded. The next lemma shows, however, that any graded left D-module M is the direct sum of D-transitive modules.

Lemma 1.3 Let D be a graded division ring and $H = \{h \in G : D_h \neq 0\}$. For any graded D-module M, then if $M_g \neq 0$, $DM_g = \bigoplus_{h \in H} D_h M_g = \bigoplus_{h \in H} M_{hg}$ is a transitive D-submodule of M. Furthermore, $M = \bigoplus_{g \in P} DM_g$, where P is a set of representatives from the cosets H_g.

Proof Since, for $h, k \in H$, $D_k(D_h M_g) = D_{kh} M_g$, DM_g is transitive. Also it is clear that $DM_g = DM_k$ if and only if $gk^{-1} \in H$; if $gk^{-1} \notin H$, then $DM_g \cap DM_k = 0$. The statement follows.

Corollary 1.4 Every homogeneous basis of a graded left D-module M has the same cardinal number. Call this cardinal number the dimension of the D-module M.

Proof Let B be a basis of M. Then if $M_g \neq 0$, $B \cap DM_g$ is a basis for

DM_g, and the statement follows.

2 Gr-primitive Rings and the Density Theorem

Definition 2.1 A G-graded ring A is called(right)gr-primitive if there exists a faithful gr-simple graded(right) A-module. A graded ideal I of A is called gr-primitive if A/I is a gr-primitive ring. If a graded right A-module M is gr-simple, then $A/\mathrm{Ann}_A(M)$ is gr-primitive and $\mathrm{Ann}_A(M)$ is a gr-primitive ideal. $J_G(A)$ (see[1,2]) is the intersection of the gr-primitive ideals of A, so that $A/J_G(A)$ is a subdirect product of gr-primitive rings.

Remark 2.2 Many of the standard results about ungraded primitive rings have analogues in the graded case. For example, if A is gr-primitive and commutative, then A is a gr-field. To see this, note that if A is gr-primitive, (0) is a modular gr-maximal right ideal in A(cf[3.p.7]). Thus A has an identity and has no nonzero proper graded right ideals; thus A is a gr-field.

The next proposition examines the structure of the graded division ring $\mathrm{END}_A(M)$, M a graded faithful gr-simple A-module.

Proposition 2.3 Suppose A is a gr-primitive ring, M a graded faithful gr-simple A-module and $D = \mathrm{END}_A(M)$. Then, if $M_g \neq 0$, $\mathrm{Hom}_{A_e}(M_g, M_{hg}) \simeq D_h$ as A_e-modules and $\mathrm{End}_{A_e}(M_g) \simeq D_e$ as rings.

Proof Suppose $M_g \neq 0$. If $M_{hg} = 0$, then since $D_h M_g \subseteq M_{hg}$, $D_h = 0$ and $\mathrm{Hom}_{A_e}(M_g, M_{hg}) \simeq D_h$. Suppose $M_{hg} \neq 0$. Let $0 \neq m \in M_g$. Then, since M is a gr-simple A-module, $M = \sum_{t \in G} mA_t$, and $M_g = mA_e$. For each ϕ in $\mathrm{Hom}_{A_e}(M_g, M_{hg})$, define $\phi' \in D_h = \mathrm{END}_t(M)_h$ by $\phi'(ma_t) = \phi(m)a_t$. We must show that ϕ' is well-defined. If $ma_t = 0$, then $ma_t A_{t-1} = 0$ and $\phi(m)a_t A_{t-1} = \phi(ma_t A_t^{-1}) = 0$. If $\phi(m)a_t \neq 0$, then $M_{hg} = \phi(m)a_t A_{t-1} = 0$, a contradiction. Thus $\phi(m)a_t = 0$ and ϕ' is well-defined. Clearly if $\phi \neq 0$, then $\phi' \neq 0$ so that the mapping $\phi \to \phi'$ from $\mathrm{Hom}_{A_e}(M_g, M_{hg})$ to D_h is one-one and it is also easy to verify that it is onto. If $h = e$, then it is straightforward to check that the mapping described above preserves multiplication.

Next we consider a graded version of the Jacobson density theorem.

Definition 2.4 (i) Let D be a graded division ring and M a graded left D-module. A graded subring A of $\mathrm{END}_D(M)$ is called gr-dense in $\mathrm{END}_D(M)$ if, for every D-linearly independent set $\{m_1, \cdots, m_n\} \subseteq h(M)$ and every set $\{y_1, \cdots, y_n\} \subseteq M$, there exists $a \in A$ such that $x_i a = y_i$ for $i = 1, \cdots, n$.

(ii) Let $D' = D_e'$ be a trivially graded division ring, and M' a G-graded left D'-module. A graded subring A' of $\mathrm{END}_{D'}(M')$ is called componentwisedense in $\mathrm{END}_{D'}(M')$ if, for all $g \in G$, for any D'-independent set $\{m_1, \cdots, m_n\} \subseteq M_g'$ and set $\{y_1, \cdots, y_n\} \subseteq M'$, there is an $a' \in A'$ such that $m_i a' = y_i \; i = 1, \cdots, n$.

Theorem 2.5 For A a G-graded ring, the following are equivalent.

i) A is gr-primitive.

ii) A is gr-dense in $\mathrm{END}_D(M)$ for D a graded division ring and M a graded left D-module.

iii) A is componentwise-dense in $\mathrm{END}_D(M)$ for D a trivially graded division ring and M a graded left D-module.

Proof The proof that (i) is equivalent to (ii) is similar to the proof for ungraded rings[3].

(ii)$\Rightarrow$(iii). If A is gr-dense in $\mathrm{END}_D(M)$, A is componentwise-dense in $\mathrm{END}_{D_e}(M)$ since a set $\{m_1, \cdots, m_n\} \subseteq M_g$ is D_e-independent if and only if it is D-independent.

(iii)$\Rightarrow$(i). If $D = D_e$ is a division ring and $A \subseteq \mathrm{END}_D(M)$ is componentwisedense, then M is a faithful gr-simple graded right A-module, so A is gr-primitive.

Although, since $J_G(A) \cap A_e = J(A_e)$ (see[4]), A a subdirect product of gr-primitive rings implies A_e is a subdirect product of primitive rings, the following example shows that A gr-primitive does not imply A_e primitive.

Example 2.6 Let k be a field and $A = M_2(k)$, the ring of 2×2 matrices over k. A is $Z/3Z$-graded by $A_0 = kE_{11} + kE_{22}$, $A_1 = kE_{12}$ and $A_2 = kE_{21}$. Let M be the 1×2 matrices over k, i.e. $M = [k, k]$ with M_0

$= [k, 0], M_1 = [0, k]$ and $M_2 = 0$. M is a faithful graded right A-module and is gr-simple; therefore A is gr-primitive. However, $A_0 \simeq k \oplus k$ is not primitive.

In the following situation, we have A_e primitive and A gr-primitive.

Proposition 2.7 Let D be a graded division ring, M a graded transitive left D-module and A a graded subring of $\mathrm{END}_D(M)$. Suppose for some nonzero M_g, A_h is dense in $\mathrm{Hom}_{D_e}(M_g, M_{gh})$ for all h. (In particular, A_e is dense in $\mathrm{End}_{D_e}(M_g)$ so that A_e is primitive.) Then $A \subseteq \mathrm{END}_D(M)$ is gr-dense, so that A is gr-primitive.

Proof Without loss of generality, suppose $M_e \neq 0$. Let $\{m_1, \cdots, m_n\}$ be a D-independent set of elements of $h(M)$ with $m_i \in M_{gi}$ and let $\{y_1, \cdots, y_n\} \subseteq M$. Since M is transitive, $D_{gi^{-1}} \neq 0$ for all i. For each i, let $0 \neq d_i \in D_{gi^{-1}}$. Then $\{d_1 m_1, \cdots, d_n m_n\} \subseteq M_e$ is D_e-independent. For all $h \in G$ with $(d_i y_i)_h \neq 0$ for some i, since A_h is dense in $\mathrm{Hom}_{D_e}(M_e, M_h)$, there exists $0 \neq a_h \in A_h$ such that $d_i m_i a_h = (d_i y_i)_h$ for $i = 1, \cdots, n$. Thus for $a = \sum a_h$, $d_i m_i a = d_i y_i$, so that $m_i a = y_i$ as required.

Proposition 2.8 A graded ring A is right gr-Artinian and gr-primitive if and only if $A \simeq \mathrm{END}_D(M) = \mathrm{End}_D(M)$ for some graded division ring D and finite dimensional left graded D-module M. In this case, A is also left gr-Artinian.

Proof Suppose A is gr-primitive and right gr-Artinian. Then A is gr-dense in $\mathrm{END}_D(M)$ for some graded division ring D and left graded D-module M. Now argue as in[5, 2.1.8]or[3, p.39]to see that M has no infinite set of homogeneous D-linearly independent elements. But if M has a finite basis and A is dense in $\mathrm{End}_D(M)$, $A = \mathrm{End}_D(M) \simeq M_n(D)$ where n is the dimension of M.

Conversely, if $A \simeq \mathrm{End}_D(M) \simeq M_n(D)$ for some graded division ring D and finite dimensional left graded D-module M, then A is a finite dimensional left and right graded module over the gr-Artinian ring. D. But then A is an epimorphic image of a gr-Artinian D-module of the form

$\bigoplus_{i=1}^{m} D(g_i)$ where, as usual, $D(g)$ denotes the g-suspension of D. Thus A is gr-Artinian.

Proposition 2.9 A graded ring A is right gr-Artinian with $J_G(A) = 0$ if and only if A is isomorphic to a finite direct sum of $M_{ni}(D_i)$, D_i a graded division ring.

Proof The proof is similar to the proof of the ungraded Wedderburn-Artin theorem(cf[3, p.40]or[6, p.153]).

Corollary 2.10 If A is right gr-Artinian with $J_G(A) = 0$, then A is gr-Artinian.

Note Some of the results in this paper were announced in [7].

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分次除环和 Jacobson 稠密性定理

Graded Division Rings and the Jacobson Density Theorem

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摘要 讨论了分次除环和分次 Jacobson 稠密性定理，证明了分次 Jacobson 根为零的右分次阿丁环也是左分次阿丁环.

关键词 分次本原环；稠密定理；左分次 Artin 环

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赋值图的张量代数的同构问题

Isomorphism Problem for Tensor Algebras Over Valued Graphs*

Abstract In this paper the isomorphism theorem for tensor algebras over valued graphs is proved and some relations between algebraic properties of tensor algebras and geometric properties of valued graphs are investigated.

Keywords valued graph, tensor algebra, noetherian algebra, prime algebra.

In [2] and [3], we have studied the isomorphism problem for path algebras over an (oriented) graph and relations between algebraic properties of these algebras and geometric properties of the underlying graphs. In this paper, we extend these results to the tensor rings over valued graphs.

An (oriented, in general, infinite) valued graph is a triple $(\sum; d, g)$, where $\sum$ is a set (of the vertices) and both d and g refer to functions defined on $\sum \times \sum$ with values which are cardinal numbers (possibly zero). We shall simply write d_{ij} and g_{ij} for $d(i, j)$ and $g(i, j)$, respectively and describe the functions graphically as follows:

$$i \underset{(g_{ji}, d_{ji})}{\overset{(d_{ij}, g_{ij})}{\rightleftarrows}} j.$$

* Project supported by the National Natural Science Foundation of China.

A collection $(D_i, {}_iM_j; i, j \in \sum)$, where D_i are division rings and ${}_iM_j$ are D_i-D_j-bimodules satisfying the conditions

$$[{}_iM_j : D_i] = d_{ij} \text{ and } [{}_iM_j : D_j] = g_{ij},$$

is said to be a modulation of the valued graph $(\sum; d, g)$.

For brevity, we shall write simply $\sum$ and (D, M) instead of $(\sum; d, g)$ and $(D_i, {}_iM_j; i, j \in \sum)$.

Recall the definition of a tensor ring $T = T(\sum, D, M)$ of a given valued graph $\sum$ with a modulation (D, M): Write $D = \bigoplus_i D_i$, $M = \bigoplus_{i,j} {}_iM_j$ and consider M as a D-D-bimodule; then

$T = D \oplus M \oplus M^{(2)} \oplus \cdots \oplus M^{(n)} \oplus \cdots$, where $M^{(n+1)} = M^{(n)} \bigotimes_D M$,

with multiplication induced by tensor products.

In the representation theory of algebras, $\sum$ is usually assumed to be finite and d_{ij}, g_{ij} are non-negative integers satisfying the conditions

$$d_i d_{ij} = g_{ij} d_j \text{ with positive integers } d_i,\ i \in \sum.$$

Under these assumptions, there is always an F-modulation of $\sum$: D_i are F-algebras with $[D_i : F] = d_i$ and F acts on ${}_iM_j$ centrally. In general situation, we shall always assume that our valued graphs admit a modulation.

The isomorphism problem is resolved by the following theorem.

Theorem 1 Let $\sum = (\sum; d, g)$ and $\sum' = (\sum'; d', g')$ be valued graphs, and suppose $(D, M) = (D_i, {}_iM_j; i, j \in \sum)$ and $(D', M') = (D_\alpha', {}_\alpha M_\beta'; \alpha, \beta \in \sum')$ are their respective modulations. Write $T = T(\sum; D, M)$ and $T' = T'(\sum'; D', M')$. If $f: T \to T'$ is a (ring) isomorphism, then there is a bijective correspondence $\theta: \sum \to \sum'$ such that for all $i, j \in \sum$,

$$D_i \simeq D'_{\theta(i)} \text{ and } {}_iM_j \simeq {}_{\theta(i)}M_{\theta(j)}.$$

Proof Let I and I' be ideals of T and T' generated by all ${}_iM_j$ with $i \neq j$ and all ${}_\alpha M_\beta'$ with $\alpha \neq \beta$, respectively. Fix $N = \oplus\, {}_iM_i$, $N' = \oplus\, {}_iM_i'$, $B = N \oplus N^{(2)} \oplus \cdots \oplus N^{(n)} \oplus \cdots$, and $B' = N' \oplus N'^{(2)} \oplus \cdots \oplus N'^{(n)}$

$\oplus \cdots$. Then, as an additive group, there hold

$$T = D \oplus B \oplus I \text{ and } T' = D' \oplus B' \oplus I'.$$

First, we claim that $f(I) = I'$. In order to prove this equality, it is clearly sufficient to show that

$$f({}_iM_j) \subseteq I' \text{ for all } i \neq j \text{ of } \sum.$$

Let $m \in {}_iM_j$, $i \neq j$ and let

$$f(m) = d' + b' + x' \text{ with } d' \in D', b \in B' \text{ and } x' \in I'.$$

Since $m^2 = 0$, we have

$$0 = f(m^2) = f^2(m) = d'^2 + (d'b' + b'd' + b''') + y' \text{ with } y' \in I',$$

and thus $d'^2 = 0$. This implies $d' = b' = 0$ and $f(m) = x' \in I'$, as required.

Consequently, the isomorphism f induces an isomorphism

$$\bar{f}; D \oplus B \simeq T/I \to D' \oplus B' \simeq T'/I'.$$

It is easy to see that every idempotent element in $D \oplus B$, in fact, belongs to D. Moreover, denote by e_i the identity element of D_i, and then $E = \{e_i; i \in \sum\}$ is the set of all primitive idempotents of D. It follows that $\bar{f}$ induces a bijection between E and the corresponding set $E' = \{e'_\alpha; \alpha \in \sum'\}$ of all identity elements e_α' of D_α'. Hence, we have established a bijection $\theta: \sum \to \sum'$. In what follows, we identify $\sum$ and $\sum'$.

Now, consider the subrings

$$T_i = e_i(D \oplus B)e_i = D_i \oplus B_i \text{ and } T_i' = e_i'(D' \oplus B')e_i' = D_i' \oplus B_i';$$

with $B_i = {}_iM_i \oplus {}_iM_i^{(2)} \oplus \cdots$, $B_i' = {}_iM_i' \oplus {}_iM_i'^{(2)} \oplus \cdots$. Obviously, $\bar{f}$ induces an isomorphism $f_i: T_i \to T_i'$. It is easy to see that the set of all invertible elements of T_i and T_i' is $D_i \setminus \{0\}$ and $D_i' \setminus \{0\}$, respectively. Hence, $\bar{f}$ induces an isomorphism

$$D_i \simeq D_i' \text{ for all } i \in \sum.$$

We identify D_i with D_i' under this isomorphism once more.

In the next step, we are going to show that ${}_iM_i \simeq {}_iM_i'$ as D_i-D_i-bimodules.

To this end, consider the above isomorphism f_i and write, for $m \in$

${}_iM_i$,

$$f_i(m) = -d_m' + m' + x' \text{ with } d_m \in D_i' = D_i, m' \in {}_iM_i' \text{ and }$$
$$x' \in {}_iM'^{(2)}_i + {}_iM'^{(3)}_i + \cdots.$$

Since $f_i(d) = d$ for $\forall\, d \in D_i$, we have

$$f_i(d_m + m) = m' + x'.$$

Let N be the additive group generated by $\{d_m + m, m \in {}_iM_i\}$. Using

$$f_i(amb) = af_i(m)b, \forall\, a, b \in D_i = D'_i, m \in {}_iM_i,$$

we can easily check that

$$d_{a_1m_1b_1+\cdots+a_sm_sb_s} = a_1 \cdot d_{m_1} \cdot b_1 + \cdots + a_s \cdot d_{m_s} \cdot b_s, \forall\, a_j, b_j, m_j.$$

This asserts that N is a D_i-D_i-module and is isomorphic to D_i-D_i-module ${}_iM_i$. Thus, for proving ${}_iM_i \simeq {}_iM_i'$ it is enough to show D_i-D_i-module $N \simeq {}_iM_i'$.

Let

$$B_N = N + N^2 + N^3 + \cdots \subseteq T_i,$$
$$B_i' = {}_iM_i' \oplus {}_iM_i'^{(2)} \oplus {}_iM_i'^{(3)} + \cdots \subseteq T_i'.$$

Since $a \cdot B_N \cdot b \subseteq B_N, a, b \in D_i$ and $(d_{m_1} + m_1)B_N(d_{m_2} + m_2) \subseteq B_N$, B_N is an ideal of T_i. Obviously, $f_i(B_N) \subseteq B_i'$, thus having $B_N \neq T_i$, and it follows $T_i = D_i \oplus B_N$ (direct sum of additive groups). Consequently $T_i'' = D_i \oplus f_i(B_N)$ (direct sum of additive groups), thus $f_i(B_N) = B_i'$, and then we have the following D_i-D_i-module isomorphisms:

$$N \simeq B_N/B_N^2 \simeq f_i(B_N)/(f_i(B_N))^2 = B_i'/B_i'^2 \simeq {}_iM'_i.$$

Our claim is proved.

Finally, we shall prove that ${}_iM_j \simeq {}_iM'_j$ as D_i-D_j-bimodules for $i \neq j$, in general. Since $f(I) = I'$, we have also $f(I^2) = I'^2$ and thus we have an isomorphism

$$\bar{\bar{f}}: D \oplus (\bigoplus_{i,j} {}_iM_j) \simeq T/I^2 \to T/I'^2 \simeq D' \oplus (\bigoplus_{i,j} {}_iM'_j).$$

We have proved $\bar{f}(e_i + I) = e_i' + I$, thus having

$$\bar{\bar{f}}(e_i + I^2) = e'_i + m'_i + I'^2 \text{ with } m_i' \in I.$$

It follows that for $i \neq j$,

$$(e_i + I^2)(I/I^2)(e_j + I^2) \simeq (e_i' + m_i' + I'^2)(I'/I'^2)(e_j' + m_j' + I'^2),$$

which yields an additive group isomorphism

$$\overset{=}{f}_{ij} : {}_iM_j - {}_iM_j'.$$

Since $\overset{=}{f}$ induces an isomorphism between D_i and $D_i'(i \in \sum')$ similarly, we conclude that $\overset{=}{f}_{ij}$ is, in fact, a bimodule isomorphism.

The proof is completed.

As corollaries we state the following two theorems.

Theorem 2 Assume that T and T' are a tensor ring of a valued graph $\sum$ and a tensor ring of a valued graph $\sum'$ respectively. Then $\sum \simeq \sum'$ if and only if $T \simeq T'$.

Theorem 3 Let $\sum$, $\sum'$ be oriented graphs and K be a field. Then the path algebra $K[\sum]$ of $\sum$ is isomorphic to path algebra $K[\sum']$ of $\sum'$ if and only if $\sum \simeq \sum'$.

Theorem 3 has been proved in [2]. The proof given here is simpler than that given in [2].

In this section, we investigate some relations between geometric properties of valued graphs and algebraic properties of their tensor rings. Using the notations of the preceding section, we may formulate the following theorems.

Theorem 4 The tensor ring $T = T(\sum; D, M)$ is left Artinian if and only if

(1) $\sum$ is finite;

(2) d_{ij} are finite for all $i, j \in \sum$;

(3) the graph has no(oriented)cycles, i.e. there is no sequence $i_1, i_2, \cdots i_n$ of elements from $\sum$ such that $d_{i_t i_{t+1}} \neq 0$ for all $1 \leqslant t \leqslant n$ with $i_{n+1} = i_1$.

Theorem 5 The tensor ring $T = T(\sum; D, M)$ is primary if and only if (1) and (3) of Theorem 4 hold.

Theorem 6 The tensor ring $T = T(\sum; D, M)$ is prime if and only

if $\sum$ is connected, i.e. if for every $i, j \in \sum$, there is a sequence of elements from $\sum$, $i = i_1, i_2, \cdots, i_n = j$ such that $d_{i_t i_{t+1}} \neq 0$ for all $1 \leqslant t \leqslant n-1$.

Theorem 7 (a) The jacobson radical $J(T)$ of T is a D-D-module generated by all regular monomials; here, a monomial $q \in T$ is said to be regular if there are (possibly trivial) monomials p_1, p_2 such that
$p_1 \otimes q \otimes p_2 = m_{i_1} \otimes m_{i_2} \otimes \cdots \otimes m_{i_n}$ where $m_{i_t} \in {}_{i_t}M_{i_{t+1}}, 1 \leqslant t \leqslant n$, with $i_{n+1} = i_1$.

(b) T is semiprimitive if and only if there are no regular monomials in T.

(c) The Bear radical of T is equal to $J(T)$.

Theorem 8 The tensor ring $T = T(\sum; D, M)$ over valued graph is left noetherian if and only if $\sum$ satisfies the following conditions:

(1) $\sum$ is finite;

(2) a_{ij} are finite for all $i, j \in \sum$;

(3) $d_{i_1 i_2} \cdot d_{i_2 i_3} \cdots d_{i_n i_1} \neq 0$ implies that $d_{i_t i_{t+1}} = 1$ and $d_{j i_{t+1}} = 0, j \neq i_t$, for all $1 \leqslant t \leqslant n$ (here, $i_{n+1} = i_1$).

Proofs of the above theorems are rather routine or similar to the corresponding ones given in [3] for path algebras. Here, we present only a proof of Theorem 8.

Proof of Theorem 8. First, it is easy to check the necessity of the conditions (1), (2) and (3). In order to prove that these conditions are sufficient, let

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$$

be an ascending chain of left ideals of T. In view of (1), $\sum_{i \in \sum} e_i = 1$ is the identity of T, and to show that the above chain stabilizes after a finite number of steps, it is sufficient to show that

$$e_i I_1 e_j \subseteq e_i I_2 e_j \subseteq \cdots \subseteq e_i I_n e_j \subseteq \cdots \quad \text{for every } i, j \in \sum \qquad (*)$$

is stabilized.

We now consider the structure of the left D_i-space e_iTe_j generated by the monomials

$p = m_{i_1} \otimes m_{i_2} \otimes \cdots \otimes m_{i_n}$ with $m_{i_1} \in {}_iM_{i_2}, m_{i_t} \in {}_{i_t}M_{i_{t+1}}$ $2 \leqslant t \leqslant n-1$, and $m_{i_n} \in {}_{i_n}M_j$.

A monomial $m_{j_1} \otimes m_{j_2} \otimes \cdots \otimes m_{j_n}$ with $m_{j_t} \in {}_{j_t}M_{j_{t+1}}$ is said to be cyclic if $j_1 = j_{n+1}$, and we will say, it is a cyclic monomial at j_1. If there are no $p \in e_iTe_j$ containing cyclic submonomials, then by (1), (2)and the fact that the left D_i-dimension of ${}_iM_j \otimes {}_jM_k$ is equal to $d_{ij}d_{jk}$, e_iTe_j is a left D_i-finite-dimensional space, i.e.

$$e_iTe_j = D_ix_1 + \cdots + D_ix_s, x_t \in e_iTe_j.$$

If there are $p \in e_iTe_j$ containing cyclic submonomials, then by(3)all those cyclic submonomials should be at i and there exists a cyclic monomial y at i such that every cyclic monomial at i in T has the form dy^m, $d \in D_i$, $m \in \mathbf{Z}^+$. Thus, in this case we have

$$e_iTe_j = D_i[y](D_ix_1 + \cdots + D_ix_s),$$

where $D_i[y]$ is the subring generated by $\{D_i, y\}$. It is easy to see that $D_i[y]$ is a left Euclidean ring. Hence the left finitely generated $D_i[y]$-module e_iTe_j is left noetherian by the generalized Hilbert finite basis theorem.

To sum up, the chain(*)in both cases is stabilized and the theorem is proved.

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路代数的张量积与有向图的直积

Tensor Products of Path Algebras and Direct Products of Directed Graphs

Let Δ, Δ' be two directed graphs and $K(\Delta), K(\Delta')$ be their path algebras over a field K. In this paper we prove that if algebras $K(\Delta)$ and $K(\Delta')$ are prime, or semiprimitive, or right noetherian, then their tensor product $K(\Delta) \otimes_K K(\Delta')$ is also of the same kind of algebras. We also discuss relations between $K(\Delta), K(\Delta')$ and $K(\Delta \times \Delta')$ and extend the above results on path algebras to the corresponding ones on tensor algebras over valued graphs.

§ 1. Preliminaries

In [1,2] we have proved the isomorphism theorem for path algebras over directed graphs and discussed relations between geometric properties of directed graphs and algebraic properties of their path algebras. In [3] we extend results of [1,2] to the corresponding ones on tensor algebras. Continuing [2,3], in this paper we study tensor products of path algebras and tensor products of tensor algebras.

A directed graph by definition is a collection $\Delta = (\Delta_0, \Delta_1, s, e)$, where Δ_0, Δ_1 are (finite or infinite) sets of vertices and of arrows respectively, vertices and arrows will be denoted by $a, b, \cdots$ and by $\alpha, \beta, \cdots$ respectively, s, e, are maps from Δ_1 to Δ_0 and $s(\alpha), e(\alpha)$ are called the source vertex and

the end vertex of α respectively. $(a \mid \alpha_1 \cdots \alpha_l \mid b)$or $\alpha_1 \cdots \alpha_l$ for short, is called a path having length $l \geqslant 1$, from vertex a to b where α_i, α_{i+1} are connected (i.e., $e(\alpha_i) = s(\alpha_{i+1})$)arrows, $a = s(\alpha_1)$and $b = e(\alpha_l)$, and suppose that $(a \mid a) = e_a$is the path from a to a with length zero. A path from a to a having length $\geqslant 1$ is said to be cyclic path. We call $\alpha_i \cdots \alpha_j$ a subpath of $\alpha_1 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_l$, where $1 \leqslant i \leqslant j \leqslant l$.

Fix field K. The path algebra $K(\Delta)$ over directed graph Δ is defined as the K-vector space with basis consisting of all possible paths of Δ with multiplication defined as following:

$$\alpha_1 \cdots \alpha_l \cdot \beta_1 \cdots \beta_m = \begin{cases} \alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_m, & \text{if} e(\alpha_l) = s(\beta_1), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $K(\Delta)$is an associative algebra in which we have, in particular, that $e_a \cdot e_a = e_a^2, e_a \cdot \alpha = \alpha$, if $s(\alpha) = a$.

Let $\Delta = (\Delta_0, \Delta_1, s, e), \Delta' = (\Delta_0', \Delta_1', s', e')$be two directed graphs having no elements in common. Construct from them a new directed graph $\Delta \times \Delta'$called direct product of Δ and Δ' in the following way:

$(\Delta \times \Delta')_0 = \{(a, a'), \forall a \in \Delta_0, \forall a' \in \Delta_0'\}$

$(\Delta \times \Delta')_1 = \{(\alpha, e_{a'}), (e_a, \alpha'), \forall a \in \Delta_0, \alpha \in \Delta_1, \alpha' \in \Delta_0', \alpha' \in \Delta_1'\}$

and define $(s(\alpha), a')$and$(e(\alpha), a')$as source vertex and end vertex of $(\alpha, e_{a'})$, $(a, s(\alpha'))$and$(a, e(\alpha'))$as source vertex and end vertex of (e_a, α'). It is easy to see that direct product of directed graphs satisfies associative law.

Proposition 1 Let Δ, Δ' be directed graphs, and K be a field, then

$$K(\Delta \times \Delta')/J \simeq K(\Delta) \otimes_K K(\Delta'),$$

where J is the ideal of $K(\Delta \times \Delta')$generated by the set of all elements

$$(\alpha_1, \alpha_1') \cdots (\alpha_n, \alpha_n') - (\beta_1, \beta_1') \cdots (\beta_m, \beta_m')$$

with $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$ and $\alpha_1' \cdots \alpha_n' = \beta_1' \cdots \beta_m'$(remark: here and in the following, α_i may be arrows or in the form e_a).

Proof Define K-linear map:

$$\theta: K(\Delta \times \Delta') \to K(\Delta) \otimes_K K(\Delta'),$$

$$(\alpha_1, \alpha_1') \cdots (\alpha_n, \alpha_n') \mapsto \alpha_1 \cdots \alpha_n \otimes \alpha_1' \cdots \alpha_n'.$$

It is easy to check that θ is a homomorphism of algebra $K(\Delta \times \Delta')$onto algebra $K(\Delta) \otimes K(\Delta')$and Ker θ is the ideal J given in our proposition.

Therefore, $K(\Delta) \otimes_K K(\Delta')$may be considered as path algebra of $\Delta \times \Delta'$with commutative relations, and this provides us an intuitive graphical understanding of algebra $K(\Delta) \times K(\Delta')$. For convenience we recall here those results of [2] which will be used in the consequence.

Theorem A(Proposition 6 in [2]). The following are equivalent:

1) Δ is a connected directed graph, i.e., any two distinct vertices are connected by a path in Δ.

2) $K(\Delta)$is a prime algebra.

Theorem B(Proposition 5 in [2])

1) The Jacobson radical $J(K(\Delta))$of $K(\Delta)$is the K-space spent by all regular paths in Δ(i.e. paths which are not subpaths of cyclic paths);

2) $K(\Delta)$is semiprimitive⇔there are no regular paths in Δ;

3) $K(\Delta)$is semiprimitive⇔$K(\Delta)$is semiprime.

Theorem C(Proposition 3 in [2]). The followings are equivalent:

(1) Directed graph Δ satisfies the following conditions:

i)$\Delta_0, \Delta_1$are finite sets; ii)for arbitrary cyclic path$(a \mid \alpha_1, \cdots, \alpha_n \mid a)$ there exists no arrow $\alpha = (a \mid \alpha \mid b) \neq \alpha_1$.

(2) $K(\Delta)$is right noetherian.

§ 2. Main Results

In this paragraph, $\Delta, \Delta', \cdots$denote directed graphs, K is an arbitrary but fixed field and tensor products of path algebras $K(\Delta)$, $K(\Delta')$ and so on are all taken relative to the ground field K, i.e. $\otimes = \otimes_K$. It is well known that tensor products of some very "good" algebras have a completely other face than their original one. But for path algebras the situation is nice. We have

Proposition 2 Let $\Delta^i, 1 \leqslant i \leqslant n$(natural number), be directed graphs, then

(a) $K(\Delta^{(i)}), 1 \leqslant i \leqslant n$, are prime algebras⇔$K(\Delta^{(1)} \times \cdots \times \Delta^{(n)})$is a

prime algebra;

(b) $K(\Delta^{(i)}), 1 \leqslant i \leqslant n$, are prime algebras$\Leftrightarrow K(\Delta^{(1)}) \otimes \cdots \otimes K(\Delta^{(n)})$ is a prime algebra.

Proof (a) By induction it is enough to prove it in the case $n = 2$. According to Theorem A it is sufficient to prove: Δ, Δ' are connected graphs if and only if $\Delta \times \Delta'$ is a connected graph. Suppose that Δ, Δ' are connected graphs and $(a, a'), (b, b')$ are arbitrary vertices in $\Delta \times \Delta'$, then there exist $(a \mid \alpha_1, \cdots, \alpha_n \mid b)$ in Δ and $(a' \mid \alpha_1', \cdots, \alpha_m' \mid b')$ in Δ', and every path of $\Delta \times \Delta'$ contained in the inverse image of $\alpha_1 \cdots \alpha_n \otimes \alpha_1' \cdots \alpha_m'$ of $K(\Delta) \otimes K(\Delta')$ in $K(\Delta \times \Delta')$ under θ (such path in $\Delta \times \Delta'$ obviously exists) connects the vertices (a, a') and (b, b'). Conversely, if $\Delta \times \Delta'$ is connected and $(\alpha_1, \alpha_1') \cdots (\alpha_n, \alpha_n')$ is a path connecting (a, a') and (b, b'), then it is easy to see that path $\alpha_1 \cdots \alpha_n$ connects a and b in Δ and path $\alpha_1' \cdots \alpha_n'$ connects a' and b' in Δ'.

(b) Again consider only the case $n = 2$, although because tensor product of two path algebras is no more a path algebra and we cannot use here the principal of induction, but it is easy to see that our proof for the case $n = 2$ can be used also for the case of arbitrary n. It is obvious that each algebra must be prime, if their tensor product is prime algebra, so we have only one thing to prove: $K(\Delta) \otimes K(\Delta')$ is prime, if $K(\Delta), K(\Delta')$ are prime.

To do this, take arbitrarily two non-zero ideals S, T, then, as is easy to see, there exist $e_a \otimes e_{a'}, e_b \otimes e_{b'}$ such that

$$S \cdot e_a \otimes e_{a'} \neq 0, e_b \otimes e_{b'} \cdot T \neq 0.$$

By Theorem A there exists a path $r = (a \mid \alpha_1, \cdots, \alpha_n \mid b)$ in Δ and a path $r' = (a' \mid \alpha_1', \cdots, \alpha_m' \mid b')$ in Δ'. We claim:

$$S \cdot (e_a \otimes e_{a'})(r \otimes r')(e_b \otimes e_{b'}) \cdot T \neq 0.$$

To prove this we take arbitrarily

$$0 \neq s \in S \cdot (e_a \otimes e_{a'}), 0 \neq t \in (e_b \otimes e_{b'}) \cdot T,$$

and write those elements in the following form:

$$s = \sum_{i \in M} k_i (r_i \otimes r_i'), (r_i, r_i') \neq (r_j, r_j'), i \neq j, i, j \in M,$$

$$t = \sum_{i \in L} f_l (t_l \otimes t_l'), (t_l, t_l') \neq (t_m, t_m'), l \neq m, l, m \in L,$$

where k_i, f_l are non-zero elements of K, r_i, t_l- paths in Δ and r_i', t_l'-paths in Δ'. We have

$$s \cdot (r \otimes r') \cdot t = \sum_{i \in M, l \in L} k_i t_l (r_i \cdot r \cdot t_l \otimes r_i' \cdot r' \cdot t_l').$$

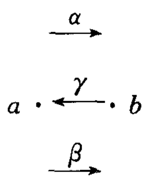
According to our assumption on s, t and the definition of multiplication of paths, we have

$$r_i \cdot r \cdot t_l \otimes r_i' \cdot r' \cdot t_l' \neq r_j \cdot r \cdot t_m \otimes r_j' \cdot r' \cdot t_m' \neq 0,$$

when$(i, l) \neq (j, m)$, consequently they are K-linear independent, but $k_i f_l \neq 0$, $\forall i \in M, l \in L$, therefore $s \cdot (r \otimes r') \cdot t \neq 0$ and, as a result, $ST \neq 0$.

After we have Proposition 2 it is natural to ask: If $K(\Delta)$is prime algebra, are path algebras of Δ with commutative relations also prime?The following example shows that this is not true.

Example Let Δ be directed graph with vertices a, b and three arrows as following:



Take commutative relations: $\alpha\gamma = \beta\gamma$, $\gamma\alpha = \gamma\beta$, then in the path algebra of Δ with those commutative relations, $K(\Delta)/I$(where I is the ideal generated by $\alpha\gamma - \beta\gamma$ and $\gamma\alpha - \gamma\beta$, the ideal S generated by$(\alpha - \beta) + I$ has property: $S^2 = 0$, this means tha $K(\Delta)/I$ is even not semiprime. But Δ is connected, consequently $K(\Delta)$ is prime. This example also says that the following Proposition 3 is not true in general for path algebras with commutative relations.

Proposition 3 Let $\Delta^{(i)}$, $i = 1, \cdots, n$, be directed graphs and K be a field.

(a) Every $K(\Delta^{(i)})$ is semiprimitive$\Leftrightarrow K(\Delta^{(1)} \times \cdots \times \Delta^{(n)})$ is semiprimitive;

(b) Every $K(\Delta^{(i)})$ is semiprimitive$\Leftrightarrow K(\Delta^{(1)}) \otimes \cdots \otimes K(\Delta^{(m)})$ is semiprimitive;

(c) $J(K(\Delta^{(1)})) \otimes \cdots \otimes K(\Delta^{(n)})$

$= \sum_i K(\Delta^{(1)}) \otimes \cdots \otimes K(\Delta^{(i-1)}) \otimes J(K(\Delta^{(i)}) \otimes K(\Delta^{(i+1)}) \otimes \cdots \otimes K(\Delta^{(n)})$.

Proof (a) By induction it is sufficient to consider the case $n = 2$ only. Discussing similarly as in Proposition 1 we know that Δ and Δ' have no regular paths(i.e. every path of positive length in $\Delta(\Delta')$ can be extended to a cyclic path)$\Leftrightarrow\Delta \times \Delta'$ has no regular paths. Then applying Th. B we get(a).

(b) Again consider only the case $n = 2$. It is easy to see our proof is all right also for general case.

Because, for arbitrary K-algebras $A, B, J(A) \otimes B$ is a quasi- regular ideal of $A \otimes B$, we have $J(A \otimes B) \supseteq J(A) \otimes B + A \otimes J(B)$ and consequently we get the part"$\Leftarrow$"of (b).

Now we are going to prove the part"$\Rightarrow$".

Suppose that the Jacobson radical J of $K(\Delta) \otimes K(\Delta')$ is not equal to zero. Take arbitrarily element x:

$$0 \neq x = \sum_i k_i x_i \in J, x_i = r_i \otimes r_i', x_i \neq x_j, i \neq j, k_i \neq 0, \quad (1)$$

where r_i, r_i' are paths of Δ and Δ' respectively. Multiplying x by elements of form $e_a \otimes e_{a'}$ from left or right when necessary we may assume that $r_i(r_i')$ have the same source vertex $a(a')$ and the same end vertex $b(b')$. By hypotheses that $K(\Delta), K(\Delta')$ are semiprimitive and by Theorem B we know that Δ and Δ' have no regular paths, therefore there exists path $r(r')$ in $\Delta(\Delta')$ which connects vertex $b(b')$ to $a(a')$. Consider

$$x \cdot (r \otimes r') = \sum_i k_i(r_i r) \otimes (r_i' r').$$

Since $x_i \neq x_j$, as $i \neq j$, so all terms appeared in the right side of the above equation are distinct, consequently, K-linear independent, therefore the element on the left side is non-zero. Note that $r_i r(r_i' r')$ are cyclic path on $a(a')$. As a result, we may assume that $r_i(r_i')$ in $x_i = r_i \otimes r_i'$ of (1) are

all cyclic path on $a(a')$. Since x is quasi-reqular element, there exists $0 \neq y \in K(\Delta) \otimes K(\Delta')$ such that

$$x + y + xy = 0, \tag{2}$$

x remains unaltered when we multiply the above equation by $e_a \otimes e_{a'}$ on the right or left sides. Therefore we may assume that

$$0 \neq y = \sum_l f_l y_l, y_l = t_l \otimes t_l', y_l \neq y_m \text{ as } l \neq m, f_l \neq 0, \tag{3}$$

where $t_l(t_l')$ are all cyclic path on vertex $a(a')$. Now (2) may be written as

$$\sum k_i(r_i \otimes r_i') + \sum_l f_l(t_l \otimes t_l') + \sum k_i f_l(r_i t_l \otimes r_i' t_l') = 0 \tag{4}$$

First of all, from the assumptions on x, y we know that all terms of (4) are non-zero and no two terms of the third sum in (4) are equal, consequently between terms of the third sum in (4) there are no K-linear relations. On the other hand, it is easy to see that $x \neq k \cdot (e_a \otimes e_{a'})$, $y \neq f \cdot (e_a \otimes e_{a'})$; as a result, in the third sum in (4) there is at least one term $r_{i_1} t_{l_1} \otimes r_{i_1}' t_{l_1}'$ which can't appear in the first and second sums in (4). Then this term can't be cancelled which contradicts the assumption that their sum equals to zero and the proof of (b) is ended.

(c) Since we have

$$J(K(\Delta) \otimes K(\Delta')) \supseteq J(K(\Delta)) \otimes K(\Delta') + K(\Delta) \otimes J(K(\Delta')),$$

it is sufficient to show that algebra

$$\begin{aligned} A &= [K(\Delta) \otimes K(\Delta')]/[J(K(\Delta)) \otimes K(\Delta') + K(\Delta) \otimes J(K(\Delta'))] \\ &\simeq [K(\Delta)/J(K(\Delta))] \otimes [K(\Delta')/J(K(\Delta'))] \end{aligned}$$

is semi-primitive. By Theorem B, $J(K(\Delta))$ is the ideal generated by all regular paths in Δ, but every regular path has a regular arrow as its subpath, so $J(K(\Delta))$ is, in fact, the ideal generated by all regular arrows in Δ. As a result, $K(\Delta)/J(K(\Delta)) \simeq K(\bar{\Delta})$, where $\bar{\Delta}$ is directed graph obtained from Δ by taking off all its regular arrows. According to (b), the tensor product of two semi-primitive path algebras is semi-primitive, so algebra A is semi-primitive and our proposition is proved.

Remark Since, by Theorem B, semi-primitivity is equivalent to semi-primeness for path algebras, Proposition 3 remains true if word "semi-primitive" is replaced by word "semi-prime" everywhere in its statement.

Proposition 4

(a) $K(\Delta^{(i)}), i = 1, \cdots, n$, are right noetherian algebra

$\Leftrightarrow K(\Delta^{(1)}) \otimes \cdots \otimes K(\Delta^{(n)})$ is right noetherian;

(b) $K(\Delta^{(1)} \times \cdots \times \Delta^{(n)})$ is right noetherian

$\Leftrightarrow K(\Delta^{(i)}), i = 1, \cdots, n$, are finite dimensional algebra. Here assume that every $\Delta^{(i)}$ contains at least one arrow.

Proof (a)"$\Leftarrow$". Suppose that one of those $K(\Delta^{(i)})$, e.g., $K(\Delta^{(1)})$ is not right noetherian and contains an infinite ascending chain of right ideals $\{I_n\}$, then, as is easy to see. $\{I_n \otimes \cdots \otimes K(\Delta^{(n)})\}$ should be a same kind chain in $K(\Delta^{(1)}) \otimes \cdots \otimes K(\Delta^{(n)})$, a contradiction to our assumption.

Now we are going to prove "$\Rightarrow$", considering only the case $n = 2$ as its proof suits also to general case. Let $K(\Delta), K(\Delta')$ be right noetherian, then Δ, Δ' have properties indicated in Theorem C, and from the proof of Proposition 3 in [2] we know that for any two vertices $a, b(a', b')$ of $\Delta(\Delta')$, $e_a \cdot K(\Delta)e_b(e_{a'} \cdot K(\Delta') \cdot e_{b'})$ is either finite dimensional space, or finitely generated module over $K[\omega](K[\omega'])$-algebra of polynomials over K in one indeterminate $\omega(\omega')$, where $\omega(\omega')$ is the shortest cyclic path at $b(b')$ in $\Delta(\Delta')$.

Suppose that $K(\Delta) \otimes K(\Delta')$ contains an ascending chain of right ideals:

$$R_1 \subseteq R_2 \subseteq \cdots \subseteq R_n \subseteq \cdots. \tag{5}$$

Since Δ, Δ' are finite directed graph, $K(\Delta) \otimes K(\Delta')$ has identity $1 = \sum e_a \otimes e_{a'}$, where a, a' take over all vertices of Δ, Δ'. Therefore, to prove that the chain (5) stops at finite steps, it is sufficient to show that for arbitrary a, a', b, b', the following chain

$$(e_a \otimes e_{a'})R_1(e_b \otimes e_{b'}) \subseteq (e_a \otimes e_{a'})R_2(e_b \otimes e_{b'}) \subseteq \cdots \tag{6}$$

stops at finite steps.

We have

$$M = (e_a \otimes e_{a'})(K(\Delta) \otimes K(\Delta'))(e_b \otimes e_{b'})$$
$$\simeq e_a K(\Delta) e_b \otimes e_{a'} K(\Delta') e_{b'}.$$

According the result about $e_a K(\Delta) e_b$ recalled above, we know that M is a finite dimensional vector space over K, or a finitely generated module over $K \otimes K[\omega'] = K[\omega']$ or $K[\omega] \otimes K = K[\omega]$-algebra of polynomials over K in one indeterminate, or a finitely generated module over $K[\omega] \otimes K[\omega']$-algebra of polynomials over K in two indeterminates. By Hilbert basis theorem. M is a right noetherian module in all those cases. Since (6) is an ascending chain of submodules in M, (6) must stop at finite steps.

(b) As easy to see, it is enough to consider the case $n = 2$ and to prove "$\Rightarrow$"only.

Suppose that $K(\Delta \times \Delta')$ is right noetherian, consequently, its homomorphic image $K(\Delta) \otimes K(\Delta')$ is the same, and by (a) we have that $K(\Delta)$ and $K(\Delta')$ are also the same. By Theorem C we have that either $K(\Delta)(K(\Delta'))$ is of finite dimension or $\Delta(\Delta')$ contains cyclic paths. Suppose. for example, that Δ has a cyclic path:

$$a \xrightarrow{\alpha} b \xrightarrow{\beta} \cdots \xrightarrow{\gamma} a.$$

Take arbitrary arrow $a' \xrightarrow{\alpha'} b'$ from Δ'. Its existence is guaranteed by our assumption. Then in $\Delta \times \Delta'$ we have:

$$\begin{array}{l} (a, a') \xrightarrow{(\alpha, e_a)} (b, a') \xrightarrow{(\beta, e_{a'})} \cdots \xrightarrow{(\gamma, e_{a'})} (a, a') \\ \quad\; \downarrow {\scriptstyle (e_a, \alpha')} \\ (a, b') \end{array}$$

According to Theorem C we conclude that $K(\Delta \times \Delta')$ is not right noetherian. This says that there are no cyclic paths in Δ and Δ'. But Δ and Δ' are finite directed graphs, as $\Delta \times \Delta'$ is finite (by Theorem C), so $K(\Delta)$ and $K(\Delta')$ are finite dimensional. Our Proposition is proved.

§ 3. Tensor Algebra Over Valued Graphs

In this section we extend results in above paragraph to the case of

tensor algebras over valued graphs.

Recall that a finite valued graph is a triplet $(\sum; d, g)$, where $\sum$ is a finite set (of vertices), its elements are denoted by $a, b, \cdots, d$ and g are maps from $\sum \times \sum$ to set of non-negative integers. Write $d(a, b) = d_{ab}$, $g(a, b) = g_{ab}$ and suppose that there exists positive integer d_a, $\forall a \in \sum$, such that $d_a d_{ab} = g_{ab} d_b$, $\forall a, b \in \sum$. Describe the maps graphically as follows:

$$a \quad \overset{(d_{ab}, g_{ab})}{\longrightarrow} \quad b$$
$$\underset{(g_{ba}, d_{ba})}{\longleftarrow}$$

An F-modulation of a valued graph $(\sum; d, g)$ is, by definition, a collection $(D_a, {}_aM_b, a, b \in \sum)$, where D_a are division algebras over a fixed field F, ${}_aM_b$ are D_a-D_b-bimodules, field F acts on ${}_aM_b$ centrally and the following equations hold: $[D_a : F] = d_a$, $[{}_aM_b : D_a] = d_{ab}$, $[{}_aM_b : D_b] = g_{ab}$, $\forall a, b \in \sum$.

For the sake of brevity we write simply $\sum$ for valued graph $(\sum; d, g)$ and (D, M, F) for its F-modulation $(D_a, {}_aM_b, a, b \in \sum)$

For an given field F, F-modulation (D, M, F) of a finite valued graph $\sum$ always exists([4]), from it we can construct a F-tensor algebra $T = T(\sum, D, M, F)$ over $\sum$ with(D, M, F)as follow: Let $D = \bigoplus_a D_a$, $M = \bigoplus_{a,b} {}_aM_b$ and M may be considered as D-D-bimodule naturally, then

$$T = D \oplus M \oplus M^{(2)} \oplus \cdots \oplus M^{(n)} \oplus \cdots$$

where $M^{(n+1)} = M^{(n)} \otimes_D M$. The multiplication in T is induced by tensor product. It is easy to see that tensor algebras over valued graphs are generalizations of path algebras over finite directed graphs. They play an important role in the representation theory of algebras.

Hence, all valued graphs concerned contain at least one $d_{ij} \neq 0$.

For sake of convenience recall some results from[3].

Theorem D(Theorem 6 in [3]). Tensor algebra $T = T\left(\sum, D, M, F\right)$ is prime$\Leftrightarrow \sum$ is connected, i.e., for any $a, b \in \sum$ there exist a chain of vertices $a = a_1, a_2, \cdots, a_n = b$ such that $d_{a_t a_{t+1}} \neq 0, 1 \leqslant t \leqslant n\text{-}1$.

Theorem E(Theorem 7 in [3]).

(a) The Jacobson radical $J(T)$ of $T = T\left(\sum, D, M, F\right)$ is a D-D-module generated by all regular monomials; here, a monomial $q \in T$ is called regular if there are (possibly trivial) monomials p_1, p_2 such that

$$p_1 \otimes q \otimes p_2 = m_{i_1} \otimes m_{i_2} \otimes \cdots \otimes m_{i_n}$$

where $m_{i_t} \in {}_{i_t}M_{i_{t+1}}, 1 \leqslant t \leqslant n$ with $i_{n+1} = i_1$.

(b) T is semiprimitive if and only if there are no regular monomials in T.

(c) The Bear radical of T is equal to $J(T)$.

Theorem F(Theorem 8 in [3]). $T = T\left(\sum, D, M, F\right)$ is right noetherian $\Leftrightarrow \sum$ satisfies the following conditions: If

$$g_{a_1 a_2} \cdot g_{a_2 a_3} \cdots g_{a_n a_1} \neq 0,$$

then

$$g_{a_t a_{t+1}} = 1, g_{a_{t+1} b} = 0, b \neq a_t \ \forall 1 \leqslant t \leqslant n$$

here $a_{n+1} = a_1$.

The following proposition is the main result of this paragraph (in the following $\otimes = \otimes_K$).

Proposition 5 Let $\sum^{(i)}, 1 \leqslant i \leqslant n$ be finite valued graphs and $(D^{(i)}, M^{(i)}, F)$ their F-modulations with fixed field F. Let $T^{(i)} = T\left(\sum^{(i)}, D^{(i)}, M^{(i)}, F\right)$ and $A = T^{(1)} \otimes \cdots \otimes T^{(n)}$. Then we have

(a) A is prime F-algebra $\Leftrightarrow$each $T^{(i)}, 1 \leqslant i \leqslant n$, is prime F-algebra;

(b) A is semiprimitive (semiprime) F-algebra$\Leftrightarrow$each $T^{(i)}, 1 \leqslant i \leqslant n$, is semiprimitive (semiprime) F-algebra;

(c) $J(A) = \sum_{i=1}^{n} T^{(1)} \otimes \cdots \otimes T^{(i-1)} \otimes J(T^{(i)}) \otimes T^{(i+1)} \otimes \cdots \otimes T^{(n)}$;

(d) A is right noetherian F-algebra$\Leftrightarrow$each $T^{(i)}$is right noetherian F-algebra.

To prove the above proposition by using Theorems D, E, F, is the same as proving the corresponding results for path algebras by using Theorems A, B, C, if we can show first that some necessary corresponding lemmas in those proofs are true. To avoid repetition, here we simply state and prove those lemmas, and as above only for the case $n = 2$.

Lemma 1 Let

$$T = T\left(\sum, D, M, F\right), T' = \left(\sum', D', M', F\right), A = T \otimes T',$$

let $e_a(e_{a'})$be identity of $D_a(D_{a'})$. Then we have

(a) If $s, t, r \in A$and

$$s \cdot (e_a \otimes e_{a'}) \neq 0, (e_a \otimes e_{a'}) \cdot t \cdot (e_b \otimes e_{b'}) \neq 0, (e_b \otimes e_{b'}) \cdot r \neq 0,$$

then

$$s \cdot (e_a \otimes e_{a'}) \cdot t \cdot (e_b \otimes e_{b'}) \cdot r \neq 0;$$

(b) Let $e = e_a \otimes e_{a'}$ · If $e \cdot z \cdot e \neq 0, z = x, y \in A$, then

$$e \cdot x \cdot e + e \cdot y \cdot e + e \cdot x \cdot e \cdot y \cdot e \neq 0.$$

We omit its proof which is similar to the one above.

Lemma 2 Let$\{a_t, t = 1,2,\cdots,n\}$be a chain of vertices in valued graph $\sum$ with property: $g_{a_t a_{t+1}} = 1, 1 \leqslant t \leqslant n$, and $a_{n+1} = a_1$. Then we have

(a) Left D_{a1}-dimension and right D_{a1}-dimension of the D_{a1}-D_{a1}-module

$$N = {}_{a_1}M_{a_2} \otimes_D \cdots \otimes_{D_{a_n}} M_{a_1}$$

contained in $T = T\left(\sum, D, M, F\right)$ are equal to 1.

(b) The F-subalgebra of T,

$$B = D_{a_1} + N + N^{(2)} + \cdots + N^{(n)} + \cdots,$$

where N as in (a), is isomorphic skew polynomial ring $D_{a_1}[x]$ in one indeterminate x over division algebra D_{a_1}(i.e., in $D_{a_1}[x]$we have $dx = xd'$ for $d, d' \in D_{a_1}$).

Proof We know that the left D_a − dimension of ${}_aM_b = d_{ab}$ and it is

right D_b − dimension = g_{ab}. It is easy to see that

$$\text{the left } D_a - \text{dimension of } {}_aM_{b\otimes D_b}{}_bM_c = d_{ab} \cdot d_{bc}$$

$$\text{the right } D_c - \text{dimension of } {}_aM_{b\otimes D_b}{}_bM_c = g_{ab} \cdot g_{bc}.$$

So according to our assumption we have

$$\text{the right } D_{a_1} - \text{dimension of } N = g_{a_1a_2}\cdots g_{a_na_1} = 1.$$

Since $d_ad_{ab} = g_{ab}d_b$, therefore

$$d_{a_1}d_{a_1a_2} \cdot d_{a_2}d_{a_2a_3} \cdot \cdots \cdot d_{a_n}d_{a_na_1} = g_{a_1a_2}d_{a_2} \cdot g_{a_2a_3}d_{a_3} \cdot \cdots \cdot g_{a_na_1}d_{a_1}.$$

Eliminating d_{a_i} from both side of above equation we get the left D_{a_1} − dimension of $N = 1$. As a result every non-zero element x of N forms a basis of both right D_{a_1} − space N and left D_{a_1} − space N, consequently we have $dx = xd'$, $d, d' \in D_{a_1}$ and $B \simeq D_{a_1}[x]$.

Lemm 3 (a) Let D be a finite-dimensional division algebra over a field F, A be right noetherian F-algebra with identity and $D[x]$ be a skew polynomial algebra in one indeterminate x over D. Then $A \otimes_F D[x]$ is also a right noetherian algebra.

(b) Let D_i be a finite-dimensional division algebra over a field F and $D_i[x_i]$ be a skew polynomial algebra in one indeterminate x_i over D, $1 \leqslant i \leqslant n$. Then their tensor product over $FD_1[x_1] \otimes_F \cdots \otimes_F D_n[x_n]$ is a right noetherian algebra.

Proof Obviously (b) is a consequence of (a). (b) is needed in the proof of (d) in Proposition 5 above. We now prove (a). For this arbitrarily take a non-zero right ideal of $A \otimes D[x]$ and prove that it is a finitely generated one.

Suppose $\{d_1, \cdots, d_n\}$ is a F-basis of D, consequently $\{d_jx^i,\ 1 \leqslant j \leqslant n,\ i$ arbitrary nonnegative integers$\}$ is a F-basis of $D[x]$ and each non-zero element r of R can be uniquely expressed as

$$r = \sum_{i,j} a_{ji} \otimes d_j x^i,\ 0 \neq a_{ji} \in A. \tag{7}$$

For fixed j the maximum value of i in $d_j x^i$ appeared in (7), i.e., s, is called to be the highest order of r respective to d_j, and a_{js} is called to be the highest coefficient of r respective to d_j.

Let $A_1 = \{0$ and highest coefficients of all non-zero elements of R respective to $d_1\}$. By multiplying elements of R from right by $a \otimes 1$ ($a \in A$, the identity $1 \in$ D) we obtain that if $b \in A_1$, then $ba \in A_1$. By multiplying elements of R from right by $1 \otimes x^i$ (the identity $1 \in A$) we get A_1 is closed under addition. So A_1 is a right ideal of A. Similarly we know that for any non-negative integer m,

$A_{1m} = \{0$ and highest coefficients of all elements of R with highest order m respective to $d_1\}$

is a right ideal of A, and

$$A_1 = \bigcup_{m=0}^{\infty} A_{1m}.$$

Since A is right noetherian, each A_{1m} is finitely generated and $A_1 = A_{1p}$ for some positive integer p. For each A_{1m}, $m \leqslant p$, take a finite generating set (of course we can suppose all those sets have the same number t of elements): $a_{1m1}, \cdots, A_{1mt}$. Obviously $\{a_{1mi}, 0 \leqslant m \leqslant p, 1 \leqslant i \leqslant t\}$ is a finite generating set of right ideal A_1.

Take an element in R having a_{1mi} as its maximum coefficient respective to d_1 and denote it by r_{1mi}. Let R_1 be the right ideal of $A \otimes D[x]$ generated by the set $\{r_{1mi}, 0 \leqslant m \leqslant p, 1 \leqslant i \leqslant t\}$. It is easy to see that for arbitrary element $r \in R$ there exists $r_1 \in R$ such that in the expression (7) of r-r_1 there are no terms of the form $a \otimes d_1 x^i, i \geqslant 0$.

Let $R' = \{$all those elements of R whose expressions (7) contain no terms of the

and

$A_2 = \{0$ and highest coefficients of all non-zero elements of R' respective to $d_2\}$,

$A_{2m} = \{0$ and highest coefficients of all elements of R' with highest order $m\}$

Although R' is no longer right ideal of $A \otimes D[x]$, using the same argument as above we still get that A_2, A_{2m} are right ideal of A. Then, repeat the above arguments and get a finite subset of

$$A \otimes D[x]: \{r_{2mi}, 0 \leqslant m \leqslant p', 1 \leqslant i \leqslant t'\}.$$

Denote by h_2, the right ideal of $A \otimes D[x]$ generated by it. Then we have that for arbitrary element $r \in R$ there exist $r_1 \in R_1$ and $r_2 \in R_2$, such that the expression (7) of r-r_1-r_2 contains no terms of the form $a \otimes d_1 x^i$ or $a \otimes d_2 x^i, i \geqslant 0$.

Continuing our argument in the above way we get $R = \sum_{i=1}^{n} R_i$, where each R_i is a finitely generated right ideal of $A \otimes D[x]$, and complete proof of our lemma.

Finally we remark that Lemma 1 is true for tensor algebras over general (infinite and d_a, d_{ab} can be infinite cardinals) valued graphs, and consequently the statements (a) (b) (c) of Propo-sition 5 are true for general case too. (cf. the corresponding theorems in [3]). Since Lemma 3 is true under condition that D is a finite -dimensional algebra over F, the statement (d) of Proposition 5 is true only when d_a, $a \in \sum$, are positive integers.

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有限维代数的表示论在中国

Representation Theory of Finite-Dimensional Algebras in China

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Representation theory of finite-dimensional algebras, which originated at the end of the 1960's, has made rapid progress in the last decades. In the mid of the 1980's Professor Shaoxue Liu began a seminar on this theory in Beijing Normal University; after almost ten year's effort, a research group in this field has been formed in China. The aim of this article is mainly to report some of the works of this group up to 1992.

During the period of developing representation theory of algebras in China, many foreign colleagues have kindly provided their help and suggestions. Some of them have given lectures in Beijing: M. Auslander, V. Dlab, D. Happel, I. Reiten, C. M. Ringel, K. W. Roggenkamp, A. Skowronski and L. Unger. We take this opportunity to thank all of them.

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1 The Structure of Auslander-Reiten Components

Let A be an Artin algebra with the Auslander-Reiten quiver $\Gamma(A)$ and the stable Auslander-Reiten quiver $\Gamma_s(A)$, $\tau = D\mathrm{Tr}$ the Auslander-Reiten translation. An indecomposable A-module M is called stable (resp. periodic) provided $\tau^n M \neq 0$ for all $n \in \mathbf{Z}$ (resp. $\tau^m M = M$ for some $m \in \mathbf{Z}$). A component of the Auslander-Reiten quiver Γ_A which consists of stable modules is called a regular component. Riedtmann [Rm] introduced the concepts of (stable) valued translation quivers. The typical examples of those quivers are respectively $\Gamma(A)$ and $\Gamma_s(A)$. Another example of stable valued translation quiver is $\mathbf{Z}\vec{\Delta}$. Note that any connected stable valued translation quiver is of the form $\mathbf{Z}\vec{\Delta}/G$, where $\vec{\Delta}$ is a connected valued oriented tree and G is an admissible automorphism group of $\mathbf{Z}\vec{\Delta}$.

If A is a representation-finite algebra over an algebraically closed field k, then Riedtmann proved in [Rm] that the components of $\Gamma_s(A)$ are of the forms $\mathbf{Z}\vec{\Delta}/G$, where Δ is a Dynkin diagram. In their paper [HPR], Happel-Preiser-Ringel proved: if C is a component of the stable Auslander-Reiten quiver of an Artin algebra containing periodic module, then C is either of the form $\mathbf{Z}\vec{\Delta}/G$, of else $\mathbf{Z}A_\infty/\langle\tau^n\rangle$, where Δ is a Dynkin diagram, and A_∞ is the infinite diagram. In fact, the existence of a periodic vertex in a connected valued stable translation quiver C implies that all vertices are periodic; in this case C is said to be periodic, otherwise non-periodic. The periodic regular components must be the regular tubes, i.e. $\mathbf{Z}A_\infty/\langle\tau^n\rangle$ for $n \geqslant 1$. The structure of periodic stable valued translation quivers with subadditive functions is also given in [HPR]. But what is the

case for the non-periodic translation quivers with subadditive functions? This was settled by Y. B. Zhang in her doctoral thesis [55] under the direction of C. M. Ringel:

Theorem([55]) Let C be a non-periodic connected valued stable translation quiver with a non-zero subadditive function f with values in N_0. Then either C is smooth and f is additive and bounded, or else $C = \mathbf{Z}\vec{\Delta}$ for some valued quiver $\vec{\Delta}$.

The proof [55] is rather complicated and technical. She considered the first homology group of the orbit graph of C and some additive function on it measuring the difference between the numbers of forward and backward arrows in any walk. In order to write C in the form $\mathbf{Z}\vec{\Delta}$, it is necessary to find a suitable orientation on the orbit graph of C. The above theorem has the following useful corollary.

Theorem([55]) Let C be a non-periodic component of the stable Auslander-Reiten quiver of an Artin algebra A. Then, $C = \mathbf{Z}\vec{\Delta}$ for some valued quiver $\vec{\Delta}$ without oriented cycles.

It is then natural to ask what kind of valued quivers $\vec{\Delta}$ can actually occur in $C = \mathbf{Z}\vec{\Delta}$, where C is a regular component of $\Gamma(A)$. If $\vec{\Delta}$ is a finite wild quiver with $|\Delta_0| \geqslant 3$, then it is well known that $H = k\vec{\Delta}$ has a regular tilting module ${}_HT$, and the connecting component of the tilted algebra $B = \mathrm{End}\ {}_HT$ is of the form $\mathbf{Z}\vec{\Delta}$ (see[R2]). Let $\vec{\Delta}$ be a connected symmetrizable valued quiver without oriented cycle and assume that after deletion of finitely many vertices and arrows one obtains a disjoint union of quivers of type A_∞; then there exists an algebra with regular component of the form $\mathbf{Z}\vec{\Delta}$ (see[CR]).

S.P. Liu [17] introduced the concepts of the left and right degrees of irreducible maps; this turned out to be useful in dealing with the possible shapes of Auslander-Reiten components. He used these concepts to give a new (non-combinatorial) proof of the Happel-Preiser-Ringel theorem which states that periodic regular Auslander-Reiten components are regular tubes; the proof does not even use Riedtmann's structure theorem for stable

translation quivers.

In [R3] Ringel proposed the following open problem: "Let A be a finite-dimensional algebra over an algebraically closed field. Is it true that all but finitely many components of $\Gamma(A)$ are of the form $\mathbf{Z}A_\infty$, $\mathbf{Z}A_\infty/\langle\tau^n\rangle$ and $\mathbf{Z}D_\infty$?" Y.B. Zhang ([56]) used the spectral radius of the Coxeter translations of some labeled trees to prove that any non-periodic regular component of the Auslander-Reiten quiver of an Artin algebra, whose growth number is less than $c = \sqrt[3]{\frac{1}{2}+\sqrt{\frac{23}{108}}} + \sqrt[3]{\frac{1}{2}-\sqrt{\frac{23}{108}}}$, is of the form $\mathbf{Z}A_\infty$, $\mathbf{Z}A_\infty^\infty$, $\mathbf{Z}B_\infty$, $\mathbf{Z}C_\infty$ or $\mathbf{Z}D_\infty$. This gives a partial solution to the problem above.

C.C. Xi ([29]) determined the structures of a finite-dimensional connected wild hereditary algebra A with the growth number $\rho(A) < c$.

Ringel [R3] has conjectured that in an Auslander-Reiten component P only finitely many indecomposable modules can have the same composition factors. Y.B. Zhang proved in [57] this is true for hereditary algebras, i.e. the indecomposable modules in any AR-component of a wild hereditary artin algebra are uniquely determined by their dimension vectors.

S.P. Liu showed in [17] that a maximal connected left (or right) stable subquiver of $\Gamma(A)$ containing some special τ-orbits is either a regular tube or can be embedded into $\mathbf{Z}A_\infty$. Liu's result is as follows:

Theorem ([17]) Let Γ be a maximal connected left stable subquiver of $\Gamma(A)$. Assume that there is a module X in Γ such that $l(\tau^n X)$ does not tend to infinity as n tends to infinity. Then either Γ is a regular tube or there is an infinite path $\cdots \to X_{i+1} \to X_i \to \cdots \to X_1 \to X_0$ in Γ with the properties that

1) The path meets each τ-orbit in Γ exactly once;

2) For any integer $i \geqslant 0$, the arrow $X_{i+1} \to X_i$ has trivial valuation in $\Gamma(A)$;

3) X_0 has exactly one immediate predecessor X_1 in Γ, and for each integer $i > 0$, X_i has exactly two immediate predecessors X_{i+1} and τX_{i-1}

in Γ.

As a consequences Liu showed that Ringel's above conjecture is also true for the components with only finitely many τ-orbits. And then he got that: a finite-dimensional algebra A over an algebraically closed field is representation-finite if and only if $\Gamma(A)$ admits only finitely many τ-orbits.

Recall that an indecomposable module M is said to be directing, provided that M does not belong to any cycle $M = M_0 \xrightarrow{f_1} M_1 \to \cdots \xrightarrow{f_r} M_r = M (r \geqslant 1)$, where M_i are all indecomposable and f_i are non-zero non-isomorphisms. A directing module has many pleasant properties. For example, an algebra A which has a sincere directing module must be a tilted algebra. (Recently this concept was generalized to arbitrary modules (not necessarily indecomposable, [HR2])).

An Auslander-Reiten component C which consists entirely of directing modules have been studied by Skowronski-Smalø [SS]: such a component C can have only finite many τ-orbits, and if in addition C is regular, then C is the connecting component of some convex subalgebra B of A with B a tilted algebra. It follows that an algebra can have only finitely many components which consists of directing modules. Peng-Xiao ([21]) and Skowronski ([Sk]) independently proved the following result.

Theorem ([21], [Sk]) The Auslander-Reiten quiver of an Artin algebra admits at most finitely many D Tr-orbits containing directing modules.

In order to give a criterion for an Auslander-Reiten component being a directing component (i.e. the component in which every indecomposable module is directing), P. Zhang [52] introduced the notion of quasi-slice, then proved that an AR-component C with a quasi-slice $\sum$ is directing if and only if the modules in $\sum$ are directing. For an A-module M, we may form the one-point extension $B = A[M]$, and we call the component of $\Gamma(B)$ which contains the new-added indecomposable projective B-module the extension component. The following theorem provides a method of constructing the new directing component:

Theorem ([52]) Let C_i be a directing component of $\Gamma(A)$ for $1 \leqslant i \leqslant m$, M a directing module (in the sense of [HR2]) with all indecomposable direct summands lying in the union $\bigcup_{i=1}^{m} C_i$. If M is not the proper predecessor in *A-mod* of any projective A-module, then the extension component of $\Gamma(B)$ is directing.

In [24] Peng considered the algebras over which the indecomposable projective modules are all directing: if A is such an algebra with an idempotent e, so is A/AeA; and A is representation-finite if and only if its Tits form is weakly positive definite. The following result was also proved in [24]: let A be a finite-dimensional algebra with directing projective and directing injective modules, C a component of $\Gamma(A)$ which contains at least one non-stable module; then C is a directing component.

Recall that an A-module M is said to be the middle of a short chain if there is some indecomposable A-module X such that $\mathrm{Hom}_A(X, M) \neq 0$ and $\mathrm{Hom}_A(M, \tau X) \neq 0$. By definition, an indecomposable A-module M is on a short cycle provided there is an indecomposable A-module N and nonzero nonisomorphisms $f: M \to N$ and $g: N \to M$. Short chains and short cycles were extensively studied by Reiten-Skowronski-Smal ø in [RSS1, RSS2] and Happel-Liu [HL]. In particular, these two concepts are the same for indecomposables, i.e. let M be indecomposable; then M is the middle of a short chain if and only if M is on a short cycle, see [HL].

2 Some Invariants of Modules

One of the basic problems in representation theory of algebras is to look for suitable invariants to determine indecomposable modules. The typical invariant of a module M is the dimension vector $\underline{\dim}\, M$ of M. Some criteria for indecomposable modules to be determined by their composition factors (i.e. if M, N are indecomposable with $\underline{\dim}\, M = \underline{\dim}\, N$, then $M \cong N$) can be found, for example, in [DR1, BS, AR1].

Now let the base field be algebraically closed. J. Y. Guo [2] considered two classes of algebras: algebras stably equivalent to representation-finite hereditary algebras and representation-finite self-injective algebras. Some

sufficient and necessary conditions for the indecomposable modules over these algebras to be determined by their composition factors were given. Two algebras Λ and Λ' are said to be stably equivalent, if the categories Λ-$\underline{\mathrm{mod}}$ and Λ'-$\underline{\mathrm{mod}}$ are equivalent, where Λ-$\underline{\mathrm{mod}}$ denotes the category which has the same objects as Λ-mod, and the morphism set is given by $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}(X, Y)/P(X, Y)$, where $P(X, Y)$ is the additive subgroup of Hom (X, Y) consisting of the maps which factors through some projective module. Let A be an algebra with the Gabriel quiver Q_A; the vertex i in Q_A is called a node provided $P(i)$ is not simple and for any pair of indecomposable $P(j)$, $P(h)$ and nonzero maps $f: P(j) \rightarrow P(i)$, $g: P(i) \rightarrow P(h)$, we have $fg = 0$. Let i be a node of Q_A; then a path in Q_A of the form $j \rightarrow i \rightarrow h$ is called an hindrance. A subquiver of Q_A consisting of arrows starting from a vertex j is called a joint of Q_A with the source of j.

Theorem([2]) Let A be an algebra stably equivalent to a representation-finite hereditary algebra. A sufficient and necessary condition for any indecomposable modules to be determined by their composition factors is that the Gabriel quiver Q_A satisfies the following conditions:

1) Two end points of a hindrance-free walk of Q_A with more than one arrow do not coincide.

2) The vertex set of two joint connected subquivers without a common arrow are not the same.

Theorem([3]) Let A be a connected representation-finite self-injective algebra with $\Gamma_s(A) \cong \mathbf{Z}\vec{\Delta}/\Pi$, where Δ is a Dynkin diagram with n vertices. Then every indecomposable A-module is determined by its composition factors if and only if A is either a local algebra or else the number of non-isomorphic indecomposable projective A-modules is larger than n.

From the above result we know that not all representation-finite self-injective algebras have the property that indecomposable modules are determined by their composition factors, so J. Xiao considered some other

more precise invariants to determine the indecomposable modules. For definitions of the Loewy factors $L\underline{\dim}M$ and the socle factors $S\underline{\dim}M$ of a module M we refer to, e.g. [37]. The stable AR quiver $\Gamma_s(A)$ of a representation-finite self-injective algebra A has the form $\mathbf{Z}\vec{\Delta}/\pi$, where Δ is a Dynkin diagram, and is called the Cartan class of A.

Theorem([37]) Let A be a self-injective algebra with the Cartan class A_n; then every indecomposable module M is determined by $L\underline{\dim}M$ and also by $S\underline{\dim}M$. Moreover, $L\underline{\dim}M$ is a $(0,1)$-matrix.

A typical example of this kind algebra is a Brauer block of a group algebra $k[G]$ with defect group a cyclic p-group, where G is a finite group and k is an algebraically, closed field of characteristic p.

The self-injective algebras of Cartan class D_n are divided into two types: two-cornered algebra and three-cornered algebra, according to the type of the configuration of $\mathbf{Z}D_n$; for notations we refer to the original paper [BLR]. For a two-cornered algebra A, Xiao [41] proved that the indecomposable A-module M is determined by $L\underline{\dim}M$ and also by $S\underline{\dim}M$.

Theorem([41]) Let A be a two-cornered algebra, X and Y two indecomposable non-simple modules; then $X \cong Y$ if and only if top$X \cong$ top$Y$, soc$X \cong$ soc$Y$.

For a three-cornered algebra, Xiao [42] also got similar results. The formulas for computing the Loewy factors and the socle factors of indecomposable modules over the representation-finite self-injective algebras with Gartan class A_n and D_n were also given in [37, 41, 42].

Riedtmann's theorem about the structure of stable Auslander-Reitan quiver of repre-sentation-finite self-injective algebra A over an algebraically closed field can be extended to an arbitrary field; in this case $\Gamma_s(A)$ is of form of $\mathbf{Z}\vec{\Delta}/\pi$, where Δ is a diagram of the classes $A_n, D_n, E_n(n = 6, 7, 8)$, B_n, C_n, F_4, G_2, . Yang-Xiao [46] determined the structure of indecomposable modules over a representation-finite self-injective algebra with Cartan class B_n and C_n over a perfect field k, and the k-species of representation-finite selfinjective algebras with Cartan class B_n or C_n. If k is an arbitrary

field, Xiao-Yang also considered this question for a self-injective algebra with Cartan class B_n or C_n; they determined the configurations of $\mathbf{Z}B_n$ and $\mathbf{Z}C_n$, and the admissible automorphism group π.

Theorem([45]) Let k be an arbitrary field and A a representation-finite selfinjective k-algebra with Cartan class B_n or C_n, and $\Gamma(A) = (\mathbf{Z}B_n)_C/\pi$ or $\Gamma(A) = (\mathbf{Z}C_n)_C/\pi$. If $\pi = (\tau^r)^{\mathbf{Z}} \cdot (2n-1)/r$, then A is a standard algebra.

Let A be a k-algebra; the trivial extension algebra $T(A) = A \ltimes D(A)$ is defined as: the addtive group of $T(A)$ is $A \oplus D(A) = A \oplus \mathrm{Hom}_k(A, k)$, and the multiplication is $(a, \varphi)(b, \psi) = (ab, a\psi + \varphi b)$, $\forall a, b \in A, \varphi, \psi \in D(A)$. One basic result about representation-finite trivial extension algebras is: if A is a basic connected finitedimensional k-algebra, then $T(A)$ is representation-finite of Cartan class Δ if and only if A is an iterated tilted algebra of Dynkin type Δ. Xiao [38] gave a sufficient and necessary conditions for a representation-finite selfinjective algebra to be a trivial extension algebra. Xiao-Zhang [44] studied the properties of indecomposable modules over the trivial extension algebra of an iterated tilted algebra.

Let A be an iterated tilted algebra of type $\vec{\Delta}$; the repetitive algebra $\hat{A}$ has the additive structure$(\bigoplus_{i\in\mathbf{Z}} A_i) \oplus (\bigoplus_{i\in\mathbf{Z}} Q_i)$ with $A_i = A, Q_i = D(A)$ for $i \in \mathbf{Z}$, whose multiplication is defined as follows.

$$(a_i, \varphi_i)_i \cdot (b_i, \psi_i)_i = (a_i b_i, a_{i+1}\psi_i + \varphi_i b_i)_i.$$

for $(a_i, \varphi_i)_i, (b_i, \psi_i)_i \in \hat{A}$. Let v be the Nakayama automorphism $\hat{A} \to \hat{A}$; then $T(A) \cong \hat{A}/v$ and v induces the Galois covering functor $\pi: \hat{A} \to T(A)$ and an automorphism of $\hat{A}$-mod. Then Happel's theorem in [H2] says that $\hat{A}$-$\underline{\mathrm{mod}} \cong D^b(A)$, and $\mathrm{T}_s(T(A)) \cong \Gamma(D^b(k\vec{\Delta}))/\langle T^2\tau\rangle$, where $\hat{A}$-$\underline{\mathrm{mod}}$ is the stable category of $\hat{A}$-mod; $B^b(A)$ is the derived category of A, and $T^2\tau$ is just the automorphism of $\hat{A}$ induced by the Nahayama functor v. We denote by π the covering functor from $\hat{A}$-mod to $T(A)$-mod induced by $\pi: \hat{A} \to T(A)$. The indecomposable $T(A)$-module M is said to be on platform, if there is $X \in \hat{A}$-mod such that $\pi(X) = M$ and X as an

object of $\hat{A}$- $\underline{\text{mod}}$ belongs to a component of form $\mathbf{Z}\vec{\Delta}$ of $\Gamma_s(\hat{A}\text{-}\underline{\text{mod}}) \cong \Gamma(D^b(k\vec{\Delta}))$.

Theorem([44]) Let $T(A)$ be the trivial extension of an iterated tilted algebra of type $\vec{\Delta}$, X a $T(A)$-module on platform; then the number of isoclasses of the $T(A)$-modules on platform which have the same dimension vector as X is at most n, where n is the number of vertices of $\vec{\Delta}$.

Theorem([44]) Let $T(A)$ be as above, X and Y the $T(A)$-modules on platform; then the followings are equivalent.

1) $X \cong Y$;

2) top $X \cong$ top Y, and soc $X \cong$ soc Y;

3) $L\underline{\dim}\ X = L\underline{\dim}\ Y$;

4) $S\underline{\dim}\ X = S\underline{\dim}\ Y$.

3　Tilting Modules and Tilted Algebras

We recall the definition of tilting module (see e.g. Happel-Ringel [HR1]). Assume that A is a finite-dimensional algebra over a field k; a module ${}_AT$ is called a tilting module provided the following conditions are satisfied: (i) the projective dimension of T is at most 1; (ii) $\mathrm{Ext}_A^1(T, T) = 0$ and (iii) there exists an exact sequence $0 \to {}_AA \to T_0 \to T_1 \to 0$, where $T_0, T_1 \in \mathrm{add}\ T$, the module class which consists of all possible direct sums of direct summands of ${}_AT$. Note that under the conditions (i)and (ii), (iii) is equivalent to (iii'): the number of the non-isomorphic indecomposable direct summands of T is just the number of simple A-modules. Note that any algebra A has at least one tilting module, i.e. ${}_AA$. And a selfinjective algebra has a unique tilting module. Assem constructed such an non-selfinjective algebra which has a unique tilting module; and Happel gave a sufficient and necessary condition for such an algebra, see [H2] Ⅲ.2.14. Lin-Peng [8] proved that A has a unique tilting module if and only if every simple A-module $S(a)$ is a direct summand of top $I(b)$, where $I(b)$ is an indecomposable injective module.

For a tilting triple (A, T, B)(i.e., T is a tilting module over A, and $B = \mathrm{End}(T)$), we may form the module classes: $J(T)$, $F(T)$, $X(T)$, and

$y(T)$(see [R1]); then $(\mathcal{J}(T), \mathcal{F}(T))$ is a torsion pair in A-mod and $(\mathcal{X}(T), y(T))$ is a torsion pair in B-mod. By definition, a tilting module T is separating provided that $(\mathcal{J}(T), \mathcal{F}(T))$ is splitting in A-mod, i.e. any indecomposable A-module either belongs to $\mathcal{J}(T)$, or to $\mathcal{F}(T)$; T is a splitting tilting module provided that $(\mathcal{X}(T), y(T))$ is splitting in B-mod. Note that if A is hereditary, then any tilting A-module is splitting. P. Zhang ([50]) proved that a tilting module T over a hereditary algebra A is separating if and only if T is a slice (in the sense of Ringel [R1]) in A-mod.

Slices provide an inner charcterization to the tilted algebras. Recall that an algebra B is called a tilted algebra if there exists a tilting triple (A, T, B) with A hereditary. This class of algebras is extensively studied by many people for their importance in the representation theory of algebras. For a tilting triple (A, T, B) with $A = k\vec{\Delta}$, if Δ is a Dynkin diagram, it is clear that B is representation-finite. If Δ is an Enclidean diagram, then B is represantation-finite if and only if T has a non-zero preprojective direct summand and a non-zero preinjective direct summand; and in other cases, B is of tame type, see [R1]. If Δ is wild, the representation type of B is not so easy to see. Kerner [K] has given a method to determine the representation type of B; in [25] Peng gave another algorithm:

Let A be a hereditary algebra, ${}_AT$ a tilting module; then there is an integer $s \geqslant 0$, a sequence of hereditary algebras $({}_0A, {}_1A, \cdots, {}_sA)$, a sequence of module $({}_0T, {}_1T, \cdots, {}_sT)$ and a sequence of non-negative integers $(m_0, m_1, \cdots, m_{s-1})$, such that

(a) ${}_0A = A, {}_0T = T, {}_iT$ is a tilting ${}_iA$-module, and ${}_sT$ has no preinjective direct summands in mod ${}_sA$,

(b) If $s > 0$, then ${}_iT = {}_iT' \oplus {}_iT''$ for $0 \leqslant i < s$, such that for any indecomposable direct summand X of ${}_iT'$, $\tau_i^{m_i}X$ is a non-zero projective ${}_iA$-module; and any indecomposable direct summand Y of ${}_iT''$, $\tau_i^{m_i}Y$ is not a project ${}_iA$-module (where τ_i is the relative Auslander-Reiten translation). Denote by e_i the idempotent of ${}_iA$ corresponding to projective ${}_iA$-module

${}_iT'$; then ${}_{i+1}A = {}_iA/\langle e_i\rangle$, ${}_{i+1}T = \tau_i^{m_i+1}\,{}_iT''$.

Theorem ([25]) With the above notations one has

(i) B is representation-finite if and only if ${}_sA = 0$ and $A_t = 0$.

(ii) B is of tame type if and only if ${}_sA$ or A_t is tame, and no one is wild.

(iii) B is of wild type if and only if ${}_sA$ or A_t is wild.

The shape of Auslander-Reiten component of a hereditary algebra A is well known, see [R1]; it has one preprojective component and one preinjective component, and all other components are quasi-serial, i.e., the regular tubes or the components of form $\mathbf{Z}A_\infty$. It is natural to ask what the shapes of components of a tilted algebra look like. It is well known that any tilted algebra has a connecting component , which is a directing component. The structure of a regular component of a tilted algebra which is not a connecting component is also known: it is also quasi-serial. Ringel [R3] asked the following question: what are the possible structures of non-regular components of tilted algebras? Strauss [S] proved that any tilted algebra has exactly one preprojective component and one preinjective component. S.P. Liu [18] gave a description of the other components of a tilted algebra by using the relative Auslander-Reiten sequence in $J[T]$ and $F[T]$, and ray insertions and coray insertions, which are introduced in [R1]. Let us recall some notions.

Let Γ be a translation quiver. A vertex x of Γ is called a ray vertex if there is an infinite sectional path

$$x = x[1] \to x[2] \to \cdots \to x[n] \to \cdots$$

in Γ, such that for each integer $i > 0$, the path $x \to \cdots \to x[i+1]$ is the only sectional path of length i in Γ which starts at x. For a ray vertex in Γ as above and integer $n > 0$, we may define a new translation quiver $\Gamma[x, n]$, and then define a translation quiver $\Gamma\,[x_0, n_1][x_1, n_1]\cdots[x_s, n_s]$ by induction, which is called a translation quiver obtained from Γ by ray insertions, see [18]. There is a dual concept of a coray vertex and coray insertions.

Theorem([18]) Let A be a hereditary connected Artin algebra. Let T be a tilting A-module and $B = \mathrm{End}_A T$, C a component of $\Gamma(B)$ other than the connecting component of B. If C is neither a preprojective component nor a preinjective component, then either C is quasi-serial or C is obtained from a quasi-serial translation quiver by ray insertions or by coray insertions.

Liu also determined the shapes of components of $\Gamma(B)$ which lie completely in $y(T)$, where (A, T, B) is a tilting triple with T a splitting tilting module.

A partial tilting module is a direct summand of a tilting module. If A is a hereditary algebra and $_AT$ a partial tilting module, then $B = \mathrm{End}_A T$ is also a tilted algebra. By definition, a concealed algebra B is an endomorphism algebra of a preprojective tilting module over an hereditary algebra. This is a tilted algebra which has two components containing a slice. P. Zhang [51] proved that: if M is a preprojective partial tilting module over a concealed algebra, then $B = \mathrm{End}\, M$ is either a tilted algebra of Dynkin type, or a concealed algebra. So one can check the representation type of B by its quadratic form. Using this one can easily prove: Let A be a tame concealed algebra, $T = T_0 \oplus T_1$ a tilting module with T_0 nonzero preprojective and T_1 regular; then $B_0 = \mathrm{End}_A T_0$ is also a tame concealed algebra. But the situation in the case of A being wild concealed is quite different: We have examples to show that B_0 can be of Dynkin type, tame and wild respectively.

P. Zhang [47, 48] has studied the structure and representation of one-point extension of a tilted algebra by so-called foremodules (the modules cogenerated by a slice module), especially by indecomposable foremodules. This class of algebras contains the class of tilted algebras as a proper subclass and also has global dimension at most 2. Let (A, T, B) be a tilting triple with A being a hereditary algebra, $_AM \in J(T)$, $_BN = \mathrm{Hom}_A(T, M)$; then $(A[M], T \oplus (M, k, e), B[N])$ is also a tilting triple; and $T = T \oplus (M, k, e)$ is a splitting tilting $A[M]$-module if and only if

$\mathrm{Hom}_A(F(T), \tau M) = 0$; and when T is a splitting module, $B[N]$ has some similar properties as a tilted algebra. Let $I(a)$ be an indecomposable injective module over a hereditary algebra A; then $A[I(a)]$ has the same representation type as A; by this result some classes of representation-finite tilted algebras with slices of wild type were constructed in [48].

4 Stable Equivalence and Triangle Equivalence

Let A be a finite-dimensional k-algebra, $C(A)$ the category of complexes over A-mod. and $K(A)$ the homotopy category of $C(A)$. Note that $K(A)$ is a triangulated category with the triangles of mapping cones. A morphism in $K(A)$: $X^{\cdot} \xrightarrow{f} y^{\cdot}$ is called a quasi-isomorphism if the induced morphism f^n: $H^n(X^{\cdot}) \to H^n(Y^{\cdot})$ is a group isomorphism for $n \in \mathbf{Z}$, where $H^n(X^{\cdot})$ denotes the n-th homology group of $X^{\cdot}$. We denote by S the set of all quasi-isomorphisms in $K(A)$; then S is a saturated multiplicative system in $K(A)$ (see [I]). Thus we can construct a fractional category $S^{-1}K(A)$; this is called the derived category of A and denoted by $D(A)$. In most cases we are interested in $D^b(A) = (S \cap K^b(A))^{-1} K^b(A)$, the derived category of bounded complexes over A-mod. If $gl.d.A < \infty$, then $D^b(A)$ is rather simple: $D^b(A) \cong K^b(A\text{-proj.})$, the homotopy category of bounded complexes over A-projective modules. If A is a locally bounded Frobenius k-algebra (i.e, there exists a complete set of orthogonal primitive idempotents $\{e_x \mid x \in I\}$ such that Ae_x and e_xA are finite dimensional over k for $x \in I$, and the projective-modules coincide with the injective modules), then the stable category $A\text{-}\underline{\mathrm{mod}}$ has a structure of triangulated category. In this way the stable category $\hat{A}\text{-}\underline{\mathrm{mod}}$ of the repetitive algebra $\hat{A}$ of a finite-dimensional k-algebra A is a triangulated category.

Happel [H1, H3] proved that for a finite dimensional algebra A, $D^b(A)$ and $\hat{A}\text{-}\underline{\mathrm{mod}}$ are triangle-equivalent if and only of A has finite global demension, and that two algebras which are tilt-equivalent are derived equivalent. Thus if A is an iterated tilted algebra of type $\vec{\Delta}$, then $D^b(A) \cong D^b(k\vec{\Delta})$; moreover Happel [H2] determined the stable Auslander-Reiten

quiver of $T(A)$. Peng-Xiao in [22,23] considered the converse of the Happel's above results. In [22], they obtained the following result.

Theorem([22]) If A is a locally bounded Frobenius algebra such that there is a triangle-equivalence A − $\underline{\text{mod}} \cong D^b(k\vec{\Delta})$, then there exists some tilted algebra A of type $\vec{\Delta}$ such that $A \cong \hat{A}$.

This result can be essentially considered as a generalization of Bretscher-Läser-Riedtmann's main result in [BLR] in the Dynkin case. Associated with the works of Hughes-Waschbüsch in [HW] and Tachickawa-Wakamatsu in [TW] , there are some interesting corollaries as follows. For any finite dimensional k-algebra B, if B is an iterated titled algebra of type $\vec{\Delta}$, then $\hat{B} \cong \hat{A}$ for some tilted algebra A of type $\vec{\Delta}$; they also got that if B is an iterated tilted algebra and e an idempotent, then eBe is also an iterated tilted algebra.

In [23], the following result is proved

Theorem([23]) Assume that A is a finite-dimension symmetric k-algebra such that A − mod $\cong T(k\vec{\Delta})$ − $\underline{\text{mod}}$; then $A \cong T(A)$ for some tilted algebra A of type $\vec{\Delta}$.

As a direct consequence, if B is an iterated tilted algebra of type $\vec{\Delta}$, then $T(B) \cong T(A)$ for some tilted algebra A of type $\vec{\Delta}$. In [52], P. Zhang determined the structure of $T(B)$ with a $Z\vec{\Delta}$-components: Let B be an iterated tilted algebra; then $T(B)$ has a $Z\vec{\Delta}$-component if and only if $T(B) \cong T(A)$, where A is a tilted algebra with the connecting component $Z\vec{\Delta}$.

Alperin and Auslander-Reiten have conjectured that algebras which are stably equivalent should have the same number of non-projective simple modules. This has been settled for representation-finite algebra (see Martinez [M1]) and for some other cases, but in general remains open. It is sufficient to verify the conjecture for self-injecture algebras ([M2]). A. P. Tang considered this conjecture for 3-nilpotent (i. e. $\text{rad}^2 A \neq 0, \text{rad}^3 A = 0$) self-injective algebras.

Let $\vec{Q}$ be a quiver, $\vec{Q}^s$ its separating quiver. Tang [28] associated a

basic connected 3-nilpolent self-injective algebra A with a class$(\vec{\Delta}, \varphi_0, \cdots, \varphi_{m-1}, I)$in the following way: let $A = k\vec{Q}/I$; then soc $A = R^2$, where $R =$ rad A, and $V = A/\text{soc } A$ is a basic connected 2-nilpotent algebra with the same Gabriel quiver as A, thus V is stably equivalent to the hereditary algebra $k\vec{Q}^s$. Let $\vec{\Delta}_0, \cdots, \vec{\Delta}_{m-1}$ be all the connected components of the separating quiver $\vec{Q}^s$; then $\vec{\Delta}_i \cong \vec{\Delta}$ or $\vec{\Delta}_i \cong \vec{\Delta}^{op}$ for $0 \leqslant i \leqslant m-1$ and some quiver $\vec{\Delta}$. Using anti-antomorphisims $\varphi_i: \vec{\Delta}_i \to \vec{\Delta}_{i+1}$, for $0 \leqslant i \leqslant m-1$, Tang recovered $\vec{Q}$ from $\vec{\Delta}$; in this sense the 3-nilpotent self-injective algebra A is determined by a class $(\vec{\Delta}, \varphi_0, \cdots, \varphi_{m-1}, I)$. If in addition $m \geqslant 3$, she verified the conjecture:

Theorem([28]) Assume that Λ is stably equivalent to the 3-nilpotent basic connected self-injective algebra A, and A is given by $(\vec{\Delta}, \varphi_0, \cdots, \varphi_{m-1}, I)$ in the above way. If $m \geqslant 3$, then Λ and A have the same number of non-projective simple modules.

5 Quasi-Hereditary Algebras

As pointed out by Cline-Parshall-Scott (e.g. see [CPS]), the quasi-hereditary algebra provides a bridge between representation theory of finite-dimensional algebras and that of Lie theory. Also, Dlab and Ringel have made an extensive studies on this class of algebras.

There are several equivalent definitions of quasi-hereditary algebras. The usual one is in terms of hereditary ideals, see [PS, DR2]. A quasi-hereditary algebra may have many ideals such that the corresponding factor algebras are not quasi-hereditary. C.C.Xi [36] showed that an algebra A with $A/\text{rad}^n A$ quasi-hereditary for some $n \geqslant 2$, is quasi-hereditary itself. Recall that a module M is called semilocal if M is a direct sum of local modules (i.e., the module has a unique maximal submodule). Lin-Xi proved in [9] that for a semilocal module with Loewy length m, then $\text{End}_A(\bigoplus_{i=1}^{m} M/N^i M)$, where N is radical of A, is quasi-hereditary. This generalizes a result in [DR3], which is the special case of $M = {}_A A$.

Dlab-Ringel [DR1] gave an example showing that the quasi-hereditary

class is not closed under tilting. Let (A, T, B) be a tilting triple. In [53] P. Zhang considered the following questions: under what conditions, is $B = \mathrm{End}_A T$ quasi-hereditary? And if A is quasi-hereditary, what properties does B possess? Let $T = \bigoplus_{i=1}^{n} T(i)$; one can construct the module $K(i)$ for $1 \leqslant i \leqslant n$ from $T(i)$. In the case of $K(i) \in G({}_A T)$, the answers to the above questions were given in [53].

Schur algebras are important quasi-hereditary algebras. Let $\sum_r$ be the symmetric group; then $V^{\otimes r}$ is a $\sum_r$-module under the permutation action, and the Schur algebra $S_k(n, r)$ can be defined as the endormorphism algebra of $k\sum_r$-module $V^{\otimes r}$. Note that if char $k = 0$, or char $k = p > r$, then $S_k(n, r)$ is semisimple.

The structure of the Schur algebras $S_k(p, p)$, for p a prime was given by C. C. Xi in [33]. Note that $S_k(n, r)$ with $n \geqslant r$ is Morita equivalent to $S_k(r, r)$([Gr]). So Xi's result can be stated in the following.

Theorem([33]) Let k be an algebraically closed field with char $k = p > 0$; then each block of the Schur algebra $S_k(n, p)$ with $n \geqslant p$ is either simple or Morita equivalent to the path algebra of

$$\circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \cdots \circ \underset{\beta_m}{\overset{\alpha_m}{\rightleftarrows}} \circ \qquad m \geqslant 1$$

module the ideal generated by

$$\alpha_i\alpha_{i+1},\ \beta_i\beta_{i+1},\ \alpha_{i+1}\beta_{i+1} - \beta_i\alpha_i,\quad 1 \leqslant i \leqslant m-1,\ \alpha_1\beta_2$$

where m depends only on p. Moreover, there is only one non-simple block.

The proof of this theorem uses the fact that $s(S(p, p)) = s(k\sum_p) + 1$, where $s(A)$ denotes the number of simple A-modules and the structure of the connected basic symmetric algrbra A with A/M quasi-hereditary where M is the socle of an indecomposable projective left ideal of A. For a non-simple connected basic algebra A, Xi [33] also gave a sufficient and necessary condition for A being symmetric with an indecomposable module M such that $\mathrm{End}_A({}_A A \oplus M)$ is quasi-hereditary.

In a recent paper [35] Xi has studied the q-Schur algebras.

Fix an ordering of simple A-modules. We denote by $\Delta(i)$ the maximal quotient of the indecomposable projective module $P(i)$ with composition factors of the form $E(j)$ with $j \leqslant i$, and $\Delta = \{\Delta(1), \cdots, \Delta(n)\}$; dually, we have $\nabla = \{\nabla(1), \cdots, \nabla(n)\}$. If $\mathfrak{X}$ is a set of modules, let $\mathfrak{F}(\mathfrak{X})$ be the set of modules which have a filtration with factors in $\mathfrak{X}$. Then A is said to be quasi-hereditary with respect to the fixed ordering provided: a) End $\Delta(i)$ is a division ring for $1 \leqslant i \leqslant n$; b) ${}_AA \in \mathfrak{F}(\Delta)$, see [DR4, R4]. Note that under the condition a), b) is equivalen to c): $\mathrm{Ext}_A^2(\Delta(i), \nabla(j)) = 0$ for $1 \leqslant i, j \leqslant n$. Let A be quasihereditary with a fixed ordering; then the intersection $\mathfrak{F}(\Delta) \cap \mathfrak{F}(\nabla) = add_A T$, where ${}_AT$ is a generalized tilting and generalized cotilting module, which is called the characteristic module of A([R4]). This module T is of particular interest; its endormorphism algebra is called the Ringel dual of A.

In [R4] Ringel proved that given an ordering of simple A-modules, A is quasi-hereditary if and only if there exists a module $T = \bigoplus_{i=1}^{n} T(i)$ with $T(i)$ indecomposable such that (i) $(\underline{\dim} T(i))_i = 1$; (ii) $\bigoplus_{j=1}^{n} T(j)$ is a generalized tilting A/J_{i+1}-module for $1 \leqslant i \leqslant n$. Note that the characteristic module of a quasi-hereditary algebra satisfies these two conditions. P. Zhang [53] proved the following.

Theorem([53]) Let A be an algebra with a fixed ordering of simple A-modules. If T is an A-module satisfing the two conditions above, then T is exactly the characteristic module of A.

6 Hall Algebras

The Hall algebra of a finitary ring has been introduced by C. M. Ringel; there is a strong relation between Hall algebras and the quantum groups, see [R5 ~ 6]. Let R be a finitary ring, $N_1, \cdots, N_t$, and M finite R-modules. Denote by $F_{N_1 \cdots N_t}^M$ the number of filtrations

$$M = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_t = 0$$

of M such that $U_{i-1}/U_i \cong N_i$ for $1 \leqslant i \leqslant t$. Let $H(R)$ be the $\mathbf{Q}$-vector space with basis $(u_{[M]})_{[M]}$ (indexed by the set of isoclasses $[M]$ of all

finite R-modules M), with a multiplication

$$u_{[N_1]}u_{[N_2]} = \sum_{[M]} F^M_{N_1 N_2} u_{[M]}$$

then $H(R)$ is an associative **Q**-algebra with $1 = u_{[0]}$. In general, $H(R)$ is *not commutative*.

Let $\vec{\Delta}$ be a cyclic quiver with n vertices, and k a field of p elements; denote by $\boldsymbol{T}$ the category of locally nilpotent finite dimensional representations of Δ over k. Then J.Y. Guo [7] determined the centre of $H(\boldsymbol{T})$.

Theorem([7]) The center of $H(\boldsymbol{T})$ is exactly the subalgebra of $H(\boldsymbol{T})$ generated by $c = \sum_{\underline{\dim} M=(1,\cdots,1)} (1-p)^{n(M)} u_{[M]}$, where $n(M)$ is the number of indecomposable summands of the module M.

In [4] Guo considered the relation between isomorphism of the species and that of the corresponding Hall algebras.

In [5,6] Guo considered the structure of the Hall algebra $H(A)$ of a cyclic serial algebra A. The Hall polynomials and the base of the Hall algebra $H_Q(A) = H(A) \otimes_Z Q$ were given. Guo also proved that the Hall algebra $H(A)$ coincides with the Loewy algebra $L(A)$ which is the subalgebra of $H(A)$ generated by the semisimple modules. $H_Q(A)$ is a filtered ring and the associated graded ring is an iterated, skew polynomial ring; thus both are Noetherian without zero divisor.

7 Uniserial Modules Over Commutative Artin Ring

It is well known in commutative algebra that a commutative Artin ring is a direct product of a finite number of commutative Artin local rings. In the following R denotes a commutative Artin local ring; then $R \cong S/\underline{a}$ for some commutative noetherian regular local ring (S, m) and some m-primary ideal $\underline{a} \subseteq m^2$. Denote by e-dim R the embedding dimension of R, i.e. the minimal number of generators of the maximal ideal of R, and assume that e-dim $R > 1$.

In his doctorial thesis [62] under the direction of M. Auslander, J.G. Luo studied uniserial R-modules and other related modules, which often provide important information about the ring itself. An uniserial R-module

U is said to be maximal provided that U is not a proper submodule of any uniserial module.

Theorem([62]) Suppose R contains an infinite field. Let $d = e\text{-}\underline{\dim} R$ and n = Loewy length of R. Then all maximal uniserial R-modules have length n if and only if $R \simeq K[[X_1, \cdots, X_d]]/(X_1, \cdots, X_d)^n$, where K is the residue field of R. In this case, R can be embedded in a direct sum of a finite number of maximal uniserial R-modules.

In case $R = K[[X_1, \cdots, X_d]]/(X_1, \cdots, X_d)^n$ for some field K, the automorphism group of R provides an effective tool; this is because that every automorphism of $K[[X_1, \cdots, X_d]]$ induces an automorphism of R. Using this technique the following was proved in [62]:

If R contains an infinite field, let U, V be nonisomorphic uniserial R-modules; then U, V belong to different components in the AR-quiver of R. And if $\ell_R(U) \geqslant 2$, then the component containing U is stable.

In case $e\text{-}\underline{\dim} R = 2$ and let U, V be maximal uniserial R-modules, one has the following.

Theorem([62]) For any nonsplit exact sequence

$$\varepsilon : 0 \to V@>f>>E@>g>>U \to 0$$

we have

(1) up to isomorphism, ε is uniquely determined by the End $(V)^{(op)}$-submodule Im $\mathrm{Hom}_R(V, g)$ of $\mathrm{Hom}_R(V, U)$.

(2) E is indecomposable.

(3) $DTr_R E = E$.

Note that the modules fixed by DTr are of particular interest because of a result of Hoshino, which states that if Λ is an Artin algebra and M is an indecomposable Λ-module with $DTrM \simeq M$, then either Λ is local Nakayama or the AR-component containing M is a homogeneous 1-tube. Luo determined the uniserial R-modules U satisfing $DTrU \simeq U$.

Denote by m the maximal ideal of S, where $R = S/a$, a criterion when E_g decomposes was given in [62], where

$$0 \to S/m \to E_g \to Tr_R(S/m) \to 0$$

is an almost split sequence. As a consequence one gets: Suppose R contains an algebraically closed field; then R is tame if and only if e-dim $R = 2$ and E_g decomposes.

8 Some Related Topics

Gabriel's theorem ([G]), which says that any basic connected finite dimensional algebra over an algebraically closed field k is of the form $A = k\vec{\Delta}/I$ for some finite quiver $\vec{\Delta}$ and some admissible ideal I of the path algebra $k\vec{\Delta}$, also provides a new point of view to some classical problems in finite-dimensional algebras, or more generally, in ring theory, in terms of quivers and relations.

In [12] Liu-Luo-Xiao considered the isomorphism problem for path algebra of quivers; in [11] Liu studied the relations between the algebraic properties of path algebras and the combinatorial properties of the corresponding quivers. These results were extended by Liu in [13] to the tensor rings over valued graphs. An (oriented, in general, infinite) valued graph is a triple $(\sum, d, g)$, where $\sum$ is a set of vertices and both d and g refer to functions defined on $\sum \times \sum$ with values which are cardinal numbers (possibly zero and ∞). A collection $(D_i, {}_iM_j, i, j \in \sum)$, where D_i is a division ring and ${}_iM_j$ a $D_i - D_j$ - bimodule satisfying the condition

$$[{}_iM_j : D_i] = d_{ij} \quad \text{and} \quad [{}_iM_j : D_j] = d_{ij}$$

is said to be a modulation of the valued graph $(\sum, d, g)$. The tensor ring $T = T(\sum, D, M)$ is defined as

$$T = D \oplus M \oplus M^{(2)} \oplus \cdots \oplus M^{(n)} \oplus \cdots$$

where $M^{(n+1)} = M^{(n)} \otimes_D M$, $D = \oplus_i D_i$, $M = \oplus_{i,j} {}_iM_j$, with the multiplication induced by tensor products. The following theorem describes the isomorphism of two tensor rings in terms of the one of the valued graphs.

Theorem([13]) Let $\sum = (\sum, d, g)$ and $\sum' = (\sum', d', g')$ be two valued graphs, and suppose that $(D, M) = (D_i, {}_iM_j, i, j \in \sum)$ and $(D', M') = (D'_\alpha, {}_\alpha M_\beta', \alpha, \beta \in \sum')$ are their modulations respective-

ly. *Write* $T = T(\sum, D, M)$ and $T' = T(\sum', D', M')$. If $f: T \to T'$ is a ring isomorphism, then there is a bijective map $\theta: \sum \to \sum'$ such that for all $i, j \in \sum$,

$$D_i \simeq D'_{\theta(i)} \quad \text{and} \quad {}_iM_j \simeq {}_{\theta(i)}M_{\theta(j)}.$$

Sufficient and necessary conditions for a tensor ring $T = T(\sum, D, M)$ to be left Artinian, prime, primary, semiprimitive were also given in [13]. The Jacobson radical $J(T)$ (which coincides with the Baer radical $B(T)$) is also calculated. We quote only the following result.

Theorem ([13]) The tensor ring $T = T(\sum, D, M)$ is left noetheian if and only if $\sum$ satisfies the following conditions

(1) $\sum$ is finite;

(2) d_{ij} are finite for $i, j \in \sum$;

(3) $d_{i_1i_2} \cdot d_{i_2i_3} \cdot \cdots \cdot d_{i_ni_1} \neq 0$ implies that $d_{i_ti_{t+1}} = 1$ and $d_{ji_{t+1}} = 0$ for $j \neq i_t$, where $1 \leqslant t \leqslant n$.

For a finite quiver $\vec{\Delta}$ (which may have oriented cycles), Xiao ([43]) proved that $k\vec{\Delta}$ is hereditary, and any (left) projective $k\vec{\Delta}$-module is a direct sum of the modules of form $(k\vec{\Delta}) \cdot e$, where e is a vertice in $\vec{\Delta}$.

We denote by $H_i(A)$ (resp. $H^i(A)$) the Hochschild homology (resp. cohomology) groups of A. Let $B = A[M]$ (resp. $B = [M]A$) be the one-point extension (resp. coextension) of A. The relation of $H^i(B)$ and $H^i(A)$ has been given by Happel([H]). Liu-Zhang proved in [15] that $H_i(B) \cong H_i(A)$ for $i > 0$ and $H_0(B) \cong H_0(A) \oplus k$. (From this one can also easily get that $H_i(A) = 0$ for $i > 0$ and $H_0(A) = k^{n(A)}$ for a triangular algebra A (i.e., an algebra whose Gabriel quiver has no oriented cycle), which was first given in [C]). We call a cycle α of length m a basic cycle if α has exactly m distinct vertices. Let $A = k\vec{\Delta}/I$ be a monomial algebra (i.e. I is generated by some paths in $\vec{\Delta}$); then $H_0(A) \cong k^{n(\vec{\Delta})}$ if and only if for any cycle $\alpha = a_1a_2\cdots a_m$, $\{\alpha, t\alpha, \cdots, t^{m-1}\alpha\} \cap I \neq \varnothing$, where a_i is an arrow for $1 \leqslant i \leqslant m$, and t is the operator defined as $t\alpha = a_2\cdots a_ma_1$

(see[15]). We call an algebra $k\vec{\Delta}/J^t(t \geq 2)$ a t-truncated algebra, where J is the ideal generated by all arrows in $\vec{\Delta}$. For integers t and n we denote by $a(n,t)$(resp. $b(n,t)$)the number $t[\frac{n}{2}]+u(n+1)$(resp. $t[\frac{n}{2}]+1-u(n)$), where $u(n)=\begin{cases}1 & n \text{ is even}\\ 0 & n \text{ is odd}\end{cases}$; then we have

Theorem([16]) Let A be a t-truncated algebra; then

1) If $\vec{\Delta}$ does not contain any cycle of length l for $a(n,t)\leq l\leq b(n,t)$, then $H_n(A)=0$.

2) If $H_n(A)=0$, then $\vec{\Delta}$ does not contain any basic cycle of length l for $a(n,t)\leq l\leq b(n,t)$.

In deformation theory of algebras, $H^2(A)=0$ is of particular interest, since this implies that A has only trivial deformation. Zhang ([54]) has determined the structures of triangular t-truncated algebras A with $H^2(A)=0$.

In the case of $t=2$, we can do some more: Let $A=k\vec{\Delta}/J^2$ be a triangular algebra, $n\geq 2$ an integer, then $H^n(A)=0$ if and only if $\vec{\Delta}$ does not contain the subquiver $\tilde{\mathbf{A}}_{1,n}$.

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广义路代数

Generalized Path Algebras

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Dedicated to Prof. Helmut Lenzing on his 60th Birthday

The description of basic algebras of finite dimension over algebraically closed fields as quotients of path algebras has been very useful in the representation theory, not only in providing a wide range of examples but mainly because of the description of the finitely generated modules over the given algebras (see, for instance[1,6]).

The purpose of these notes is to look at a generalization of the concept of path algebras. Instead of assigning to each vertex of a given quiver the base field K, we shall assign a K-algebra. With this, we lose the uniqueness of the quiver associated to a given algebra but we hope to get a better insight in some properties of the ring structure, for instance primeness and

noetherianness. Some related problems can be found in [2,3,4,5].

These notes are organized as follows. In section 1, after giving the definition of the generalized path algebras we prove some preliminary results on them. Section 2 is devoted to characterising generalized path algebras which are prime and noetherian, while in the last section we look at the so-called isomorphism problem for generalized path algebras.

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1 Preliminaries

This section is devoted to define generalized path algebras and to establish some preliminary results on them. Along this paper, K will denote a fixed field. For an algebra, we mean an associative K-algebra.

A *quiver* Δ is given by two sets Δ_0 and Δ_1 together with two maps s, $e:\Delta_1 \to \Delta_0$. The elements of Δ_0 are called *vertices*, while the elements of Δ_1 are called *arrows*. For an arrow $\alpha \in \Delta_1$, the vertex $s(\alpha)$ is the *start vertex of* α and the vertex $e(\alpha)$ is the *end vertex of* α, and we draw $s(\alpha) \xrightarrow{\alpha} e(\alpha)$. A *path in* Δ is $(a \mid \alpha_1 \cdots \alpha_n \mid b)$, where $\alpha_i \in \Delta_1$, for $i = 1, \cdots, n$, and $s(\alpha_1) = a$, $e(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \cdots, n-1$, and $e(\alpha_n) = b$. The *length of a path* is the number of arrows in it. to each arrow α we can assign an edge $\bar{\alpha}$ where the orientation is forgotten. A *walk* between two vertices a and b is given by$(a \mid \bar{\alpha}_1 \cdots \bar{\alpha}_n \mid b)$, where $a \in \{s(\alpha_1), e(\alpha_1)\}$, $b \in \{s(\alpha_n), e(\alpha_n)\}$, and for each $i = 1, \cdots, n-1$, $\{s(\alpha_i), e(\alpha_i)\} \cap \{s(\alpha_{i+1}), e(\alpha_{i+1})\} \neq \varnothing$. A quiver is said to be *connected* if for each pair of vertices a and b, there exists a walk between them.

Let $\Delta = (\Delta_0, \Delta_1)$ be a quiver and $\mathscr{A} = \{A_i : i \in \Delta_0\}$ be a family of K-algebras A_i with identity, indexed by the vertices of Δ. Unless otherwise

stated, we shall indicated the identity of A_i as e_i, for $i \in \Delta_0$. The elements of $\bigcup_{i\in\Delta_0} A_i$ are called the *$\mathscr{A}$-paths of length zero*, and for each $n \geqslant 1$, an *$\mathscr{A}$-path of length n* is given by $a_1\beta_1 a_2\beta_2 \cdots a_n\beta_n a_{n+1}$, where $(s(\beta_1) \mid \beta_1 \cdots \beta_n \mid e(\beta_n))$is a path in Δ of length n, for each $i = 1, \cdots, n$, $a_i \in A_{s(\beta_i)}$, and $a_{n+1} \in A_{e(\beta_n)}$. Consider now the quotient R of the K-vector space with basis the set of all $\mathscr{A}$-paths of Δ by the subspace generated by all the elements of the form $(a_1\beta_1 \cdots \beta_{j-1}(a_j^1 + \cdots + a_j^m)\beta_j a_{j+1} \cdots \beta_n a_{n+1}) - \sum_{l=1}^{m} a_1(\beta_1 \cdots \beta_{j-1} a_j^l \beta_j \cdots \beta_n a_{n+1})$where $(s(\beta_1) \mid \beta_1 \cdots \beta_n \mid e(\beta_n))$is a path in Δ of length n, for each $i = 1, \cdots, n$, $a_i \in A_{s(\beta_i)}$, $a_{n+1} \in A_{e(\beta_n)}$, and $a_j^l \in A_{s(\beta_j)}$for $l = 1, \cdots, m$. Define now in R the following multiplication. Given two elements $[a_1\beta_1 \cdots \beta_n a_{n+1}]$and $[b_1\gamma_1 \cdots \gamma_m b_{m+1}]$, we define

$$[a_1\beta_1 \cdots \beta_n a_{n+1}] \cdot [b_1\gamma_1 \cdots \gamma_m b_{m+1}]$$

to be equal to $[a_1\beta_1 \cdots \beta_n (a_{n+1} b_1) \gamma_1 \cdots \gamma_m b_{m+1}]$, if a_{n+1} and b_1 belong to the same A_i and equal to 0 otherwise. It is easy to check that the above multiplication in R is well-defined and gives to R an structure of K-algebra. Also, R has identity if and only if Δ_0 is finite. The algebra R defined above is called the *$\mathscr{A}$-path algebra of* Δ and we denote it by $R = K(\Delta, \mathscr{A})$.

Remark 1.1 (i)Observe that if $A_i = K$ for each $i \in \Delta_0$, then the algebra $K(\Delta, \mathscr{A})$ defined above is the usual path algebra $K\Delta$ of Δ.

(ii)Any K-algebra R with identity can be *realized* as an $\mathscr{A}$-path algebra $K(\Delta, \mathscr{A})$by just taking Δ as the quiver consisting of a unique vertex and $\mathscr{A} = \{R\}$. Also, it is not difficult to see that a *realization* of a K-algebra as $\mathscr{A}$-path algebra is not necessarily unique. We shall discuss the problem of uniqueness in section 3 below.

(iii)Let $R = K(\Delta, \mathscr{A})$. We leave to the reader the verification that $\dim_K R < \infty$ if and only if $\dim_K A_i < \infty$ for each $i \in \Delta_0$, and Δ is a finite quiver without oriented cycles.

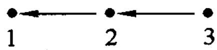
We can give an alternative definition for an $\mathscr{A}$-path algebra $R = K(\Delta, \mathscr{A})$as follows. Let ${}_iM_j$ be the free A_i-A_j-bimodule with free generators given by the arrows from i to j. if $A = \bigoplus_{i\in\Delta_0} A_i$, then ${}_iM_j$ is also an A-A-bimodule

by defining $A_k \cdot {}_iM_j = 0$ if $k \neq i$ and ${}_iM_j \cdot A_k = 0$ if $k \neq j$. Let $M = \bigoplus_{i \to j} {}_iM_j$, *which is clearly an* A-A-bimodule. We leave to the reader to check that R is isomorphic to the algebra

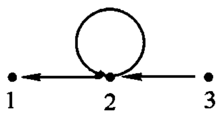
$$A \oplus M \oplus (M \otimes_K M) \oplus (M \otimes_K M \otimes_K M) \oplus \cdots$$

with multiplication given by the tensor product.

Examples 1.2 Let Δ be the quiver



and let $\mathscr{A} = \{K, K[x], K\}$. Observe that the algebra $R = K(\Delta, \mathscr{A})$ is isomorphic to the (usual) path algebra $K\Delta'$, where Δ' is the quiver



We finish this section characterizing the Jacobson radical $J(R)$ of an $\mathscr{A}$-path algebra $R = K(\Delta, \mathscr{A})$. We shall need the following definition.

Definition. Let Δ be a quiver and $\mathscr{A}$ be a family of K-algebras $\{A_i\}_{i \in \Delta_0}$.

(i) We say that a path $(i \mid \beta_1 \cdots \beta_n \mid j)$, $n \geqslant 1$, in Δ is *regular* provided it is not a subpath of an oriented cycle in Δ.

(ii) Let $(i \mid \beta_1 \cdots \beta_n \mid j)$, $n \geqslant 1$, be a regular path in Δ. Then, any $\mathscr{A}$-path $a\beta_1 \cdots \beta_n b$, with $a \in A_i$ and $b \in A_j$ is called *regular*. Also, if there are no oriented cycles passing through a vertex i, then the elements of $J(A_i)$ are called *regular $\mathscr{A}$-paths (of length zero)*.

Proposition 1.3 Let $R = K(\Delta, \mathscr{A})$ be the $\mathscr{A}$-path algebra of Δ. Then the Jacobson radical $J(R)$ of R is just the K-subspace of R generated by all regular paths of R.

Proof: Denote by M the K-subspace of R generated by all regular paths of R. Also, for $X \subset R$, denote by (X) the (bilateral) ideal generated by X. Let $u \in R$ be a regular path of length greater than zero. Clearly, $(u)^2 = 0$ and so $(u) \subset J(R)$. Let now i be a vertex at which there are no oriented cycles and consider $U_i = \sum_u (u)$, where u runs over all regular paths of length greater than zero in R, starting or ending at i.

Observe that $(J(A_i)) = J(A_i) + U_i$. On the other hand, U_i is a

quasi-regular ideal of R and since$(J(A_i) + U_i)/U_i \cong J(A_i)$ is also quasi-regular, we infer that $(J(A_i))$is quasi-regular. Hence $(J(A_i)) \subset J(R)$, and this shows that $M \subset J(R)$.

Suppose that $M \neq J(R)$. Then there exists a nonzero element $x = \sum_i p_i \in J(R)$, where each p_i is not regular. Clearly, there are l, m such that $0 \neq e_l x e_m \in J(R)$. So, without loss of generality we can assume that $e_l p_i e_m = p_i$, for each i. By hypothesis, the p_i's are subpaths of oriented cycles, and so there exists a path $q = e_m q e_l$ from m to l. Multiplying x by q, we still get a sum of nonzero paths in $J(R)$which are clearly oriented cycles. Therefore, we can assume that each p_i is an oriented cycle at the vertex l. Moreover, multiplying x on both sides by a path$(e_l \mid \gamma \cdots \delta \mid e_l)$, we may finally assume that $0 \neq x = \sum_i p_i$, where all p_i's have the form$(e_l \mid \gamma \cdots \delta \mid e_l)$. Observe that if z is a path in R with $e_l z \neq 0$, then $p_i z \neq 0$ and so x is a quasi-regular element in R. Hence, there exists $y \in R$ such that $x + y + xy = 0(*)$. Multiplying$(*)$on both side by e_l, we may suppose $e_l y e_l = y$. But now$(*)$leads to a contradiction because $xy \neq 0$ and has a bigger length than $x + y$. Therefore, $M = J(R)$, as required.

The next result is an easy consequence of the above proposition.

Corollary 1.4 Let $R = K(\Delta, \mathscr{A})$ be the $\mathscr{A}$-path algebra of Δ. Then, R is semiprimitive if and only if Δ has no regular paths, and $J(A_i) = 0$ whenever i is an isolated vertex.

2 Criteria for Primeness and Noetherianness

Along this section, let us assume that Δ is a quiver, $\mathscr{A} = \{A_i : i \in \Delta_0\}$ is a family of K-algebras A_i indexed by the vertices of Δ and with identity e_i, and $R = K(\Delta, \mathscr{A})$ is the $\mathscr{A}$-path algebra of Δ. We shall give now criteria for R to be prime and to be right and left noetherian in terms of the quiver Δ and the family $\mathscr{A}$. We start with a definition.

Definition. We say that a quiver Δ is *oriented connected* provided for any pair of distinct vertices i and j, there exists a path in Δ from i to j.

Clearly, a connected quiver Δ is oriented connected if and only if any

path of Δ is a subpath of an oriented cycle.

Theorem 2.1 With the above notations and assuming that Δ is connected and has at least 2 vertices, the following are equivalent:

(a) R is prime;

(b) Δ is oriented connected;

(c) No $\mathscr{A}$-path is regular.

Proof: $(a)\Rightarrow(b)$. Suppose Δ is not oriented connected. So, there are distinct vertices i and j with no paths from i to j. Hence, there is no $\mathscr{A}$-path in R from i to j, which implies that $e_iRe_j = 0$, a contradiction.

$(b)\Rightarrow(a)$. Let I and J be two nonzero ideals of R. Observe that there are vertices $i, j, l, m \in \Delta_0$ such that $e_iIe_j \neq 0$ and $e_lJe_m \neq 0$. Assuming now that Δ is oriented connected we infer that there exists a path $\gamma \in \Delta$ from j to l. Therefore, $e_iIe_j\gamma e_lJe_m \neq 0$, and so R is prime as required.

$(b)\Rightarrow(c)$. It is enough to show that any path in Δ is non-regular and each vertex belongs to an oriented cycle. Let $(i \mid \alpha_1\cdots\alpha_n \mid j)$ be a path in Δ with $i \neq j$. Since Δ is oriented connected, then there exists a path $(j \mid \beta_1\cdots\beta_m \mid i)$ and so $(i \mid \alpha_1\cdots\alpha_n\beta_1\cdots\beta_m \mid i)$ is a cycle. Since $|\Delta_0| \geqslant 2$, the same argument can be used to show that each vertex belongs to an oriented cycle.

$(c)\Rightarrow(b)$. Since Δ is connected, given two distinct vertices i and j, there always exists a walk between them. Denote by $l = l(i,j)$ the length of a shortest walk between i and j. We shall prove the statement by induction on $l \geqslant 1$. Suppose that $l = 1$, that is, that there exists either an arrow $\alpha: i \to j$ or an arrow $\beta: j \to i$. There is nothing to show in the former case, so let us consider the later. By hypothesis, there exists an oriented cycle $(j \mid \beta\beta_1\cdots\beta_m \mid j)$ and so $(i \mid \beta_1\cdots\beta_m \mid j)$ is the required path from i to j. Suppose now $l > 1$. So there exists a walk $(i \mid \overline{\gamma_1}\cdots\overline{\gamma_l} \mid j)$ of length l between i and j. If $s(\gamma_1) = i$, then there exists a walk $(e(\gamma_1) \mid \overline{\gamma_2}\cdots\overline{\gamma_l} \mid j)$ of length $l-1$ and so, by induction, there exists a path from $e(\gamma_1)$ to j which composed on the left with γ_1 gives a path from i to j. If now $e(\gamma_1) = i$, then the same argument used above, gives a path from i to $s(\gamma_1)$. The induction hypothesis yields a path from $s(\gamma_1)$ to j, leading to

the required path from i to j.

We shall now prove a criterion for R to be right noetherian.

Theorem 2.2 With the above notations, the algebra R is right noetherian if and only if the following conditions hold:

(i) Δ is finite.

(ii) There exists only one arrow starting at each vertex lying in an oriented cycle of Δ.

(iii) The algebra A_i is one dimensional for each vertex i lying in an oriented cycle.

(iv) If j is not a sink vertex, then A_j is finite dimensional, and if j is a sink vertex, then the algebra A_j is right noetherian.

Proof: Let us assume first that R is right noetherian. Clearly, Δ is finite, and so we have (i).

Suppose now that Δ has a cycle γ starting and ending at a vertex i and that there is an arrow $\alpha : i \to j$ which does not belong to γ. To get a contradiction, just consider the following infinite ascending chain of distinct right ideals

$$(\gamma\alpha) \subset (\gamma\alpha, \gamma^2\alpha) \subset (\gamma\alpha, \gamma^2\alpha, \gamma^3\alpha) \subset \cdots$$

This proves (ii).

Let now γ be an oriented cycle in Δ passing through a vertex i, and consider the arrows α and β such that $s(\alpha) = i$ and $e(\beta) = i$. If A_i is not one-dimensional, then there exist two elements K-linearly independent a and b in A_i. Consider $x = e_i\alpha\cdots\beta a$ and $y = e_i\alpha\cdots\beta b$. Then,

$$(xy) \subset (xy, x^2y) \subset (xy, x^2y, x^3y) \subset \cdots$$

is an infinite ascending chain of distinct right ideals, which proves (iii).

If j is a sink vertex, then every right ideal of A_j is also a right ideal of R, so A_j must be right noetherian. If j is not a sink vertex, then there exists an arrow $\alpha : j \to l$ starting at j. If now A_j has infinite dimension, let $\{a_i : i \in N\}$ be a linearly independent infinite set of elements of A_j. Then,

$$(a_1\alpha) \subset (a_1\alpha, a_2\alpha) \subset (a_1\alpha, a_2\alpha, a_3\alpha) \subset \cdots$$

gives a contradiction to the fact that R is right noetherian.

So, (i)-(iv)are necessary for R to be right noetherian.

Let us now assume that(i)-(iv)hold and prove that R is right noetherian. To do so, let

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset \cdots$$

be an ascending chain of right ideals of R. Since Δ_0 is finite, say $|\Delta_0| = m$, then $\sum_{i=1}^{m} e_i$ gives the identity of R. So it is enough to show that

$$e_iI_1e_j \subset e_iI_2e_j \subset \cdots \subset e_iI_ne_j \cdots \qquad (*)$$

stops after a finite number of steps for each pair (i,j). Clearly, for each n, $e_iI_ne_j \subset e_iRe_j$. Let γ be a path in e_iRe_j. First assume that γ does not belong to an oriented cycle. Therefore, using(ii), γ has no subpaths which are oriented cycles. If furthermore, j is not a sink vertex, then by hypotehsis e_iRe_j is a finite dimensional K-space and so $(*)$stops after a finite number of steps. In case j is a sink vertex, then e_iRe_j is a finitely generated right A_j-module. Since, by hypothesis, A_j is right noetherian, we infer using the Hilbert basis theorem, that the chain$(*)$should stop.

Suppose now j belongs to an oriented cycle. By(ii), there exists a unique basic cycle θ at j and the end vertices of paths starting at j should belong to θ. In this case, e_iRe_j is a finite generated right $K[\theta]$-module. Using now the wellknown fact that a finitely generated right R-module over a right noetherian ring R is noetherian, we infer that the chain$(*)$should stop.

It is not difficult to dualize the above result to get the following characterization of left noetherian $\mathscr{A}$-path algebras.

Theorem 2.3 With the above notations, the algebra R is left noetherian if and only if the following conditions hold:

(i)Δ is finite.

(ii)There exists only one arrow ending at each vertex lying in an oriented cycle of Δ.

(iii)The algebra A_i is one dimensional for each vertex i lying in an oriented cycle.

(iv) If j is not a source vertex, then A_j is finite dimensional, and if j is a source vertex, then the algebra A_j is left noetherian.

As a consequence, we get the following result.

Corollary 2.4 Let $R = K(\Delta, \mathscr{A})$ be an $\mathscr{A}$-path algebra. Then R is noetherian if and only if Δ is finite and for each connected component Γ of Δ one of the following facts happens:

(a) $\Gamma = \{i\}$ is an isolated vertex and A_i is noetherian;

(b) Γ has no oriented cycles and A_i is finite dimensional for each vertex i;

(c) Γ is a basic cycle and A_i is one-dimensional for each vertex i in Γ (so that the corresponding ring direct summand of R is of type $K\widetilde{\boldsymbol{A}}_n$ with $\widetilde{\boldsymbol{A}}_n$ linearly ordered).

3 The Isomorphism Problem

In this section, we shall study the so-called *isomorphism problem* for the generalized path algebras, as defined in the first section. In other words, let Δ and Γ be two quivers, and $\mathscr{A} = \{A_i : i \in \Delta_0\}$ and $\mathscr{B} = \{B_j : j \in \Gamma_0\}$ be two families of K-algebras with identities indexed by the vertices of Δ and Γ, respectively. The isomorphism problem can be posed as follows: if Φ: $K(\Delta, \mathscr{A}) \to K(\Gamma, \mathscr{B})$ is an algebra isomorphism, at which conditions do there exist a quiver isomorphism $\psi : \Delta \to \Gamma$ and an isomorphism $A_a \xrightarrow{\cong} B_{\psi(a)}$, for each $a \in \Delta_0$?

For the classical case of path algebras, that is, when $A_i \cong K$ for each $i \in \Delta_0$, and $B_j \cong K$ for each $j \in \Gamma_0$, it is well-known that an isomorphism of algebras $K\Delta \xrightarrow{\cong} K\Gamma$ induces an isomorphism of quivers $\Delta \xrightarrow{\cong} \Gamma$. On the other hand, any K-algebra can be realized as a generalized path algebra of a quiver consisting of a unique vertex. We shall look in the sequel at some situations at which the isomorphism theorem holds. We keep the notation from the previous paragraph.

From now on, let us assume that Δ and Γ are finite quivers with no oriented cycles. Assume, furthermore, that, for each $a \in \Delta_0$, the identity e_a of A_a is the unique nonzero idempotent of A_a, and, for each $b \in \Gamma_0$, the

identity f_b of B_b is the unique nonzero idempotent of B_b. Denote $R = K(\Delta, \mathscr{A})$ and $S = K(\Gamma, \mathscr{B})$ and assume that $\Phi: R \to S$ is an algebra isomorphism. For convenience, we shall consider both R and S as $\mathbf{Z}^+$-graded algebras as follows: $R = R_0 + R_1 + \cdots$ and $S = S_0 + S_1 + \cdots$, where $R_0 = \bigoplus_{a\in\Delta_0} A_a$, R_i = ($\mathscr{A}$-paths of length i in Δ), $S_0 = \bigoplus_{b\in\Gamma_0} B_b$, S_i = ($\mathscr{B}$-paths of length i in Γ). Assume, furthermore, that $|\Delta_0| = n$, $|\Gamma_0| = m$. By our hypothesis, we infer that $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$ are complete systems of primitive and pairwise orthogonal idempotents for R and S, respectively. In particular, $\sum_{i\in\Delta_0} e_i = 1_R$ and $\sum_{i\in\Gamma_0} f_i = 1_S$ are the identities of R and S, respectively.

Observe that due to the imposed conditions on the algebras of A, if Δ has no arrows, then R_0 has exactly $2^n - 1$ distinct idempotents. It should be also clear that if Δ has an arrow $\alpha: a \to b$ which is not a loop, then $e_a + x_a\alpha x_b$ is also an idempotent for each $x_a \in A_a$ and $x_b \in A_b$. Actually, if e is an idempotent in R, then $e = r_0 + r_1 + \cdots$, where r_0 is an idempotent in R_0, that is, r_0 is a sum of some e_a's. Since Φ is an isomorphism, we have $1_S = \Phi(1_R) = \Phi(e_1) + \cdots + \Phi(e_n)$, and so $\Phi(e_i) = f_{\sigma(i)} + s_{\sigma(i)}$, where $s_{\sigma(i)} \in S_1 + S_2 + \cdots$. Since $\{f_j : j \in \Gamma_0\}$ is a complete set of primitive and pairwise orthogonal idempotents, and rearranging the terms if necessary, we infer that $\Phi(e_i) = f_i + s_{(i)}$, with $s_{(i)} \in S_1 + S_2 + \cdots$, and so $m = n$. In particular, Φ induces an one-to-one correspondence between Δ_0 and Γ_0.

Lemma 3.1 Let e be an idempotent of R and $f = e + r_{10} + r_{01} + r_{00}$, where $r_{10} + r_{01} + r_{00} \in R_1 + R_2 + \cdots$, $e \cdot r_{00} = 0 = r_{00} \cdot e$, $e \cdot r_{10} = r_{10}$ and $r_{01} \cdot e = r_{01}$. Then

(a) f is an idempotent if and only if $r_{00} = r_{01} \cdot r_{10}$ and $r_{00} \cdot r_{00} = r_{10} \cdot r_{00} = r_{00} \cdot r_{01} = 0$.

(b) Suppose f is an idempotent, then f is primitive if and only if e is primitive.

Proof: (a) Observe that since Δ has no oriented cycles, then $r_{10} \cdot r_{01} = 0$. Therefore,

$(f)^2 = (e + r_{10} + r_{01} + r_{00})^2 = e + r_{10} + r_{10} \cdot r_{00} + r_{01} + r_{01} \cdot r_{10} + r_{00} \cdot r_{01} + r_{00} \cdot r_{00}$ If f is an idempotent, then $r_{10} = r_{10} + r_{10} \cdot r_{00}$, $r_{01} = r_{01} + r_{00} \cdot r_{01}$, and $r_{00} = r_{00} \cdot r_{00} + r_{01} r_{10}$(*). Clearly now, $r_{10} \cdot r_{00} = 0 = r_{00} \cdot r_{01}$ and, by(*), we get $r_{00} \cdot r_{00} = (r_{00} \cdot r_{00} + r_{01} + r_{10}) \cdot r_{00} = r_{00} \cdot r_{00} \cdot r_{00}$. Iterating this procedure, we get for each n, that $(r_{00})^2 = (r_{00})^3 = \cdots = (r_{00})^n$. Therefore, by length, we infer that $(r_{00})^2 = 0$ and so $r_{00} = r_{01} \cdot r_{10}$. The converse is clear.

(b) Suppose $e = e_2 + e_3$, where e_2 and e_3 are orthogonal idempotents. Writing $r_{10} = r_{20} + r_{30}$ with $e_2 \cdot r_{20} = r_{20}$, $e_3 \cdot r_{30} = r_{30}$ and $r_{01} = r_{02} + r_{03}$ with $r_{02} \cdot e_2 = r_{02}$, $r_{03} \cdot e_3 = r_{03}$, we will have $f = (e_2 + e_3) + (r_{20} + r_{30}) + (r_{02} + r_{03}) + (r_{02} + r_{03}) \cdot (r_{20} + r_{30}) = (e_2 + r_{20} + r_{02} + r_{02} \cdot r_{20}) + (e_3 + r_{30} + r_{03} + r_{03} \cdot r_{30})$, and so e is primitive if and only if f is primitive.

Corollary 3.2 The ideal generated by $\Phi(e_i) S \Phi(e_i)$ equals the one generated by $f_i S f_i$. In particular, $(B_i) = (\Phi(e_i) S \Phi(e_i))$.

We shall now state and prove our results concerning the isomorphism problem. Keeping the above notation, we have just seen that $|\Delta_0| = |\Gamma_0|$, and $\Phi(A_i) = B_i$, for each i.

Theorem 3.3 Suppose neither Δ nor Γ have multiple arrows. Then the isomorphism $\Phi: K(\Delta, \mathscr{A}) \to K(\Gamma, \mathscr{B})$ induces an isomorphism $\Psi: \Delta \to \Gamma$ of quivers such that for each $a \in \Delta_0$, $A_a \cong B_{\Psi(a)}$.

Proof: We shall prove it by induction on $|\Delta_0|$. If $|\Delta_0| = 1$, then $\Delta_1 = \varnothing = \Gamma_1$, because neither Δ nor Γ have oriented cycles. Also, by the above $\Phi(A_a) = B_a$, where a is the unique vertex of Δ (identifying Δ_0 with Γ_0). Suppose now that $|\Delta_0| = n > 1$. Let I_i be the ideal of R generated by A_i, and J_i the ideal of S generated by B_i. Since $\Phi(A_i) = B_i$, we have that $\Phi(I_i) = J_i$ and so Φ induces an isomorphism $\Phi': R/I_i \to S/J_i$. Writing $\hat{R}_i = R/I_i$ and $\hat{S}_i = S/J_i$, we have clearly that $\hat{R}_i = K(\Delta \setminus \{i\}, \mathscr{A} \setminus \{A_i\})$ and $\hat{S}_i = K(\Gamma \setminus \{i\}, \mathscr{B} \setminus \{B_i\})$. Using the induction hypothesis, we have that $\Delta \setminus \{i\}$ is isomorphic to $\Gamma \setminus \{i\}$, and $A_j \cong B_j$, for each $j \neq i$. Since this can be done for each i, and using the fact that neither Δ nor Γ have multiple arrows, we infer that the quivers Δ and Γ are isomorphic and, $A_j \cong B_j$, for

each j as required.

For the next result we shall have the same hypothesis on the families $\mathscr{A}$ and $\mathscr{B}$ but Δ and Γ are arbitrary finite quivers.

Theorem 3.4 Assume furthermore that the algebras A_i, $i \in \Delta_0$, B_j, $j \in \Gamma_0$, are finite dimensional. Then the isomorphism $\Phi: K(\Delta, \mathscr{A}) \to K(\Gamma, \mathscr{B})$ induces an isomorphism $\Psi: \Delta \to \Gamma$ of quivers such that for each $a \in \Delta_0$, $A_a \cong B_{\Psi(a)}$.

Proof:By counting dimensions over K, one can prove the result for $|\Delta_0| = |\Gamma_0| = 2$. By induction, using the same argument as before, we get the general case.

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附录

Appendix

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论文和著作目录

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后　记

Postscript by the Chief Editor

在搜集编写《北京师范大学数学系史》的过程中，我深深地感到，我们数学系的5位老先生：王世强，孙永生，严士健，王梓坤，刘绍学教授：

(1) 他们是我系的学术带头人，为我系的学科建设和人才培养，花费了毕生的精力，做出了重大的贡献。5位先生均为1981年被批准为首批博士生导师，王梓坤教授是1984年来到我校任校长。其余4位博士生导师，占全国首批数学博士生导师数量的1/18，占我校首批博士生导师(理科还有黄祖洽院士，刘若庄院士，陈光旭教授，汪堃仁院士和周廷儒院士)数量的2/9。数学系在学校的多种指标中一般占1/10或更低，此次批准我系的博士生导师数量，提高了我系在学校中的地位，且此举为数学系在全国数学界的地位奠定了重要基础，开创了近20多年来的良好局面。

(2) 这5位中国现代数学家，均入选《中国现代数学家传》(江苏教育出版社，第二、三、五卷，1995，1998，2002)，他们为我国现代数学的发展做出了重要的贡献。

(3) 5位先生中，在王梓坤教授来我校之前，其余4位在数学系被戏称为"四大金刚"，这4位教授均毕业于北京师范大学数学系，且终身在数学系任教，这也是数学系的特殊现象。

(4) 5位先生中后4位都出生于1929年，这是中国现代数学家在一所高校数学系独特的现象。

由此，我产生了编辑5位老先生文集的想法，并和系里的一些教师谈论过此事。《北京师范大学数学系史》出版后，与数学系主任郑学安教授

和数学与数学教育研究所所长刘永平教授商量，由我主编这套文集，并与北京师范大学出版社当时的总编马新国教授协商同意后，将5位老先生文集列入出版社的《教授文库》。

在2003年底，出版社的新一届领导将5位先生的文集列入《北京师范大学数学家文集》出版计划。经过协商，文选的内容包括：照片；序；论文选；全部论文和著作目录；后记。作为刘老师的学生，我能够非常荣幸地协助敬爱的老师在他75岁生日的时候编辑这部文集，并祝愿刘老师健康长寿。

本文集承蒙Ringel教授为其写序，其内容是中德代数合作史的极好刻画，这使文集增辉不少，我们对其表示深深的感谢。自序为刘先生发表在《中国现代数学家传》第五卷的自传，收录时做了个别字的改动。原自传的主要论著目录本文集未附，其引用的文献改为本文集附录中的序号。根据现在的情况，在自序中加了几个脚注。

《走向代数表示论》是这套文集的第五部。本书的出版得到了北京师范大学出版社的大力支持，以及刘平同志的热情帮助，在此表示由衷的感谢。

借此机会，将这一段的工作做一汇报。

在百年校庆前，在数学系的系所党总支联席会上，讨论校庆活动时，数学系主任郑学安教授建议，让我负责出一本纪念册，内容类似于像他家里有的北京大学数学系80周年系庆编的《北京大学数学系成立八十周年纪念册》。当时，我明确表示不同意出类似纪念册的宣传品。我认为，作为中国一所著名大学的重要系所，应该出一部正式出版的《系史》。要让我搞，我就主编一部《系史》出版。后在全系教师及校友的大力支持下，我就搞出一部《系史》。

可以这样说，为了在北京师范大学百年校庆前出版《北京师范大学数学系史》，除了正常的教学，带研究生，以及行政管理工作外，我中断了与我的研究方向有关的一切问题，包括编辑部退修的稿件。在搜集编写《系史》的过程中，由于原始材料都是本人亲自查阅，再加上平时搜集的数学科学学院的有关材料，使我考虑如何系统地搜集和整理我院的历史资料，在可能的情况下发表或由出版社正式出版。

作为中国一所著名大学的重要院所，有很多资料值得搜集并进行研究。《系史》出版后，我成为一位《系史》的业余研究者。除了修改《系史》

(主要是在 1952 年以前的部分内容)外,在工作之余,我决定抽出时间,做以下事情:

(1) 整理一个完整的院校调整后的北京师范大学数学科学学院本科生入学名单。

(2) 对数学科学学院的老先生做一些系列访谈。初步确定的第一批人选为 10 人。

(3) 主编并出版王世强,孙永生,严士健,王梓坤,刘绍学教授的文集。

(4) 主编并出版傅种孙, 钟善基,丁尔陞,曹才翰教授的数学教育文选。

(5) ……

这里要说的是,以上 4 项工作已基本完成。王世强、孙永生、严士健、王梓坤、刘绍学的文集由北京师范大学出版社出版;傅种孙、钟善基、丁尔陞、曹才翰的数学教育文选由人民教育出版社出版。

最后,谈谈主编 9 部文集的一点体会。出版文集最好是老先生们逢五逢十的诞辰年份出版。文集整理中最容易出现问题的是作者全部论文目录的收集,这不是一件简单的工作,且有可能耗费大量的时间和精力。此类工作宜在出版文集的作者健在时进行。即使文集作者能够提供论文目录,也应进行多方面的核实和补充。在平时,整理者如果能够养成积累所收集对象的论文目录的习惯则更好。我真切地希望将来主编《* * * 文集》作者的弟子们,平时有人能够积累导师的论著目录。我也希望每一个写论文的人,有一个自己的全部论文的目录。如果文集作者已经逝世,搜集全部论文目录的时间宜持续较长的时间,不可匆忙拿出。著作目录的搜集相对容易。

出版北京师范大学数学科学学院(系)史,将著名数学家、数学教育家和科学史专家文集进行整理和编辑出版,是我们数学科学学院学科建设的一项重要的和基础性的工作。数年后,这项工作还应继续做下去。当然,搜集和积累数学科学学院史料的工作还在继续进行。

主编　李仲来

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